ELECTRIC CIRCUITS

THEORY AND APPLICATIONS

DAHL

SHORT-CIRCUIT CALCULATIONS AND STEADY-STATE THEORY

McGRAW-HILL BOOK COMPANY

~~J. W. Gallop.~~

April 1939.

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AND

STEADY-STATE THEORY

BY

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PREFACE

The purpose of this treatise is to present the engineering aspects of circuit theory. Although theoretical, the language and viewpoint of the book are those of the engineer. It gives the methods and tools necessary for the analysis of modern power-circuit problems.

The book has grown out of the author's experience as a teacher and an electrical engineer. It is primarily intended as a textbook for the course "Electric Circuits" included in the postgraduate curriculum of the Electrical Engineering Department of the Massachusetts Institute of Technology. There has been a long-felt need for such a text. It is hoped, however, that the book also may prove of use to students of similar courses at other institutions, to electrical-engineering students in general, and to many practising electrical engineers.

Numerous illustrative examples are worked out in the text. These examples are, wherever practicable, based on actual engineering data and are representative of the type of problems with which the electrical engineer to-day deals.

The author wishes to express his appreciation and thanks to Mr. G. H. Arapakis, Instructor in Electrical Engineering at the Massachusetts Institute of Technology, who read the manuscript, offered many valuable suggestions, and who worked out and checked several of the numerical problems. Thanks are also due to Dr. E. A. Guillemin, Instructor in Electrical Engineering, for reading and criticizing parts of the manuscript, to Messrs. E. Bramhall, L. A. Bingham, C. V. Bullen, O. W. Walter, R. B. Wright, and D. S. Young, formerly graduate students of electrical engineering, for calculating the numerical data used in the example in the chapter on synchronous-machine charts, and to Mr. H. F. Goodwin of the engineering staff of Jackson and Moreland, Engineers, Boston, Mass., for preparing some of the more important drawings. Last, but not least, the writer is indebted to the authors of those technical papers and manufacturers' publications which have served as sources of material during the preparation of the manuscript.

O. G. C. Dahl.

Cambridge, Mass.,
July, 1928.

CONTENTS

SHORT-CIRCUIT CALCULATIONS
AND
STEADY-STATE THEORY

CHAPTER I

CALCULATION OF SHORT-CIRCUIT CURRENTS IN NETWORKS

There are, in general, two problems in network solution with which the electrical engineer has to deal, namely, (1) the determination of voltages, currents, and power for normal operating conditions and (2) the determination of voltages and currents during short circuits.

Analytical solutions of complicated alternating-current networks may be extremely laborious. This is particularly so when a network is fed by several generating stations and supplies numerous loads. Three methods of attack are available: (*a*) solution by simultaneous equations, (*b*) solution by trial and error, (*c*) solution by simplification of the network. In the following, the first two methods will be only briefly indicated. For a more thorough treatment, the reader is referred to other texts[1] and to the numerous papers[2] on the subject in the technical press. The method of solution by simplification will be discussed

[1] Herzog, J. and C. Feldmann, "Die Berechnung Elektrischer Leitungsnetze in Theorie und Praxis," Julius Springer, Berlin, 1921.

Woodruff, L. F. "Principles of Electric Power Transmission and Distribution, Chap. XIV, John Wiley and Sons, Inc., New York, 1925.

[2] See, for instance:

Woodward, W. R., R. D. Evans, and C. L. Fortescue, "Calculating Short-circuit Currents in Networks," *Elec. Jour.*, p. 344, 1919. This article is divided into three parts as follows:

1. "Testing with Miniature Networks," by Woodward.
2. "Analytical Solutions," by Evans.
3. "Development of Analytical Solutions," by Fortescue.

The methods presented in the last two parts are applicable also under normal operating conditions.

in more detail, as it is almost universally applied in practical short-circuit calculations.

The first method involves the setting up of a system of vector equations by applying Ohm's and Kirchhoff's laws. Upon having established the equations, the unknowns are determined by simultaneous solutions. If the number of equations is large, this process is an extremely cumbersome one, although the process of elimination may be shortened by the use of determinants.

In order to use the second method by which the correct solution is approached by steps, it is necessary to assume values of current or power in one or more branches of the network. It is impossible definitely to outline the procedure to be followed without reference to some specific problem. The calculations, however, would as a rule, be in conformance with the following general scheme:

Making use of the assumed values, calculations are carried out between points at which certain electrical conditions are definitely known. This sometimes (particularly in simple loop circuits) involves coming back to the starting point. If the electrical quantities calculated at the points at which conditions are actually known check with the latter, a correct solution is obtained. This shows, then, that the original assumptions in regard to values of current or power (or both) are correct. If, on the other hand, the known electrical conditions are not checked, the original assumptions are obviously in error. The discrepancies between calculated and known conditions may this time be used as a guide in correcting the initially assumed values of current or power, and the calculations repeated. If, upon recomputation, discrepancies between calculated quantities and known quantities still are present, a second adjustment of the assumed values with a subsequent recomputation is necessary.

Thomälen, A., "Zur zeichnerischen Behandlung beliebiger Leitungsnetze," *Elektrotek. Z.*, p. 694, 1921.

Chapman, F. T., "The Calculation of Direct-current and Alternating-current Networks," *Elec. Rev.* (London), p. 486, 1923.

Blake, D. K., "Alternating-current Secondary Networks," *General Elec. Rev.*, p. 391, 1923.

Evans, R. D., "The Analytical Solution of Networks," *Elec. Jour.*, pp. 149 and 207, 1924.

Richter, H., "Evolution of Alternating-current Secondary Networks," *Elec. Jour.*, p. 320, 1925.

Chrustschoff, W., "Beitrag zur Berechnung elektrischer Leitungsnetze." *Elektrotek. Z.*, p. 1405, 1927.

By proceeding in this manner, a correct solution is eventually reached. With some practice in the trial-and-error method, solutions sufficiently accurate for engineering purposes will, as a rule, be obtained after one or two recalculations.

The two methods just described are general and applicable to any type of network, independent of whether the network is fed by one or more generating stations and supplies one or more loads. The third method, involving simplification of the network itself, however, is somewhat limited in its application. It is generally applicable only when the network is fed by a single generating station and supplies a single load. The solution by this method is obtained by reducing the network between the generating station and the load to a single equivalent impedance. This can always be done even if the network is complicated.

So far, the discussion has been devoted to the solution of networks for normal operating conditions. When it is desired to determine voltages and currents during short circuits, the same general methods may be applied.[1] Usually, however, the solution for short-circuit conditions is a good deal simpler than for normal conditions.

The determination of short-circuit currents is a frequently recurring problem in electrical engineering. It is necessary to know the short-circuit currents at various points in the network so that the proper size of circuit breakers may be selected. During times of short circuit, the circuit breakers involved are called upon to stand the maximum short-circuit current which will flow and to interrupt the current after the elapse of a certain time. The currents which the breakers have to interrupt are usually a good deal smaller than the short-circuit currents which they carry initially. Even so, however, these currents may be extremely high and easily equal to many times the current which the breaker normally handles. The operation of many of the protective relays used in networks today also depends upon short-circuit currents. Hence, in order to enable the engineer to select and

[1] An excellent treatise on the general matter of short-circuit currents in networks is "Überströme in Hochspannungsanlagen," by J. Biermanns, Julius Springer, Berlin, 1926.

The following papers discuss special short-circuit problems:

Bekku, S., "Calculation of Short-circuit Ground Currents on Three-phase Power Networks," *Gen. Elec. Rev.*, p. 472, 1925.

Lewis, W. W., "Single-phase Short-circuit Calculations," *Gen. Elec. Rev.*, p. 479, 1925.

install the proper relays and to determine their setting, knowledge of the values of the short-circuit currents is necessary.

Since, as already mentioned, short-circuit calculations have to be performed frequently, the methods should be as direct and simple as possible. It is not essential to obtain the short-circuit currents with extreme accuracy. If it were attempted to obtain entirely rigorous solutions, the methods would become exceedingly complicated and require an undue amount of time, to say the least. The fact really is that it would be impossible in most cases to obtain rigorous solutions. Experience has shown that the approximate methods which are in common use are sufficiently accurate for engineering purposes. These methods have become standardized. They are simple and readily applicable even to complicated networks.

When a short circuit occurs, the network will, as a rule, be supplying power to several loads. The first assumption which is made, in order to facilitate the short-circuit calculations, is that the load currents may be ignored in comparison with the short-circuit currents. As a rule, the short circuit is confined to a single point only at a time, and this is always assumed in the calculations. It is further assumed that the impedance of the short circuit itself is zero, *i.e.*, a short circuit is always considered to be a "dead short circuit" in the true sense of the word.

If the network on which the short circuit occurs were supplied by one generating station only, it is obvious that the method of simplification would be immediately applicable. In order to extend the applicability of this method also to the general case where several stations supply the network, it is assumed that the induced voltages at the various generating stations are equal both in magnitude and phase. Of course, this will not be quite true, but the error introduced by this assumption should not be a serious one. When the assumption is made, however, that the generator voltages are equal, all generators supplying the system may be assumed connected to a hypothetical bus at which the common voltage is maintained. The network may then be reduced to a single equivalent impedance between the hypothetical bus and the point of short circuit, and the solution hence obtained by the method of simplification.

In reducing the network, the impedance of lines, transformers, and generators should all be included. It has become customary to neglect resistance, leakance, and capacitance of lines and

feeders and to consider their reactance only. Similarly, the resistance of transformers and generators is neglected. The transformers are represented by their equivalent reactance, their exciting currents being ignored. The reactance assigned to the generators will be discussed in more detail below. Since the entire network contains reactances only, the handling of complex quantities is avoided during the process of simplification, a fact which evidently reduces the amount of labor in no small degree.

Simplification of Networks.—There are three transformations available which may be used during the process of reducing a network to a single equivalent impedance between two points. Usually it is necessary to simplify the network by steps through a repeated application of transformations. The three methods of transformation are given below.

Δ-Y Transformation.—Whenever three impedances form a Δ or a three-cornered mesh, it is possible to convert this circuit into a Y or a three-cornered star.[1] The two circuits (Fig. 1) will be equivalent as far as conditions at the terminals are concerned.

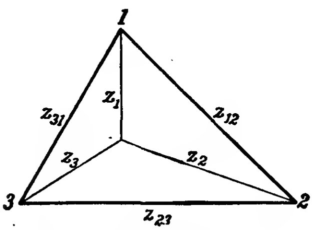

FIG. 1.—Equivalent Y- and Δ-impedances.

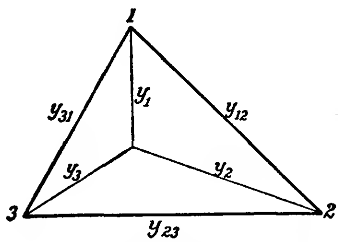

FIG. 2.—Equivalent Y- and Δ-admittances.

This means that the two circuits will offer identical impedances between any pair of terminals and will, for the same applied voltage, absorb the same amount of active and reactive power.

In terms of impedances, the conversion formulas are as follows:

$$Z_1 = \frac{Z_{31}Z_{12}}{Z_{12} + Z_{23} + Z_{31}} = \frac{Z_{31}Z_{12}}{\Sigma Z} \tag{1}$$

$$Z_2 = \frac{Z_{12}Z_{23}}{Z_{12} + Z_{23} + Z_{31}} = \frac{Z_{12}Z_{23}}{\Sigma Z} \tag{2}$$

$$Z_3 = \frac{Z_{23}Z_{31}}{Z_{12} + Z_{23} + Z_{31}} = \frac{Z_{23}Z_{31}}{\Sigma Z} \tag{3}$$

[1] KENNELLY, A. E., "The Equivalence of Triangles and Three-pointed Stars in Conducting Networks," *Elec. World and Eng.*, Vol. XXXIV, p. 413, 1899.

In terms of admittances (see Fig. 2), the conversion formulas become

$$y_1 = \frac{y_{12}y_{23} + y_{23}y_{31} + y_{31}y_{12}}{y_{23}} = \frac{N}{y_{23}} \tag{4}$$

$$y_2 = \frac{y_{12}y_{23} + y_{23}y_{31} + y_{31}y_{12}}{y_{31}} = \frac{N}{y_{31}} \tag{5}$$

$$y_3 = \frac{y_{12}y_{23} + y_{23}y_{31} + y_{31}y_{12}}{y_{12}} = \frac{N}{y_{12}} \tag{6}$$

Y-Δ Transformation.—If the given circuit is a Y or three-cornered star, it may be changed into an equivalent Δ or three-cornered mesh. In terms of impedances (refer to Fig. 1), the conversion formulas are

$$Z_{12} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} = \frac{N}{Z_3} \tag{7}$$

$$Z_{23} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1} = \frac{N}{Z_1} \tag{8}$$

$$Z_{31} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2} = \frac{N}{Z_2} \tag{9}$$

The conversion formulas in terms of admittances become

$$y_{12} = \frac{y_1y_2}{y_1 + y_2 + y_3} = \frac{y_1y_2}{\Sigma y} \tag{10}$$

$$y_{23} = \frac{y_2y_3}{y_1 + y_2 + y_3} = \frac{y_2y_3}{\Sigma y} \tag{11}$$

$$y_{31} = \frac{y_3y_1}{y_1 + y_2 + y_3} = \frac{y_3y_1}{\Sigma y} \tag{12}$$

Star-mesh Transformation.—Any star circuit may be converted into its equivalent mesh circuit independent of the number of rays in the originally given star. The converse theorem does not hold, *i.e.*, it is not, in general, possible to convert a general mesh circuit into its equivalent star.[1]

Figure 3 shows a star with n rays and its equivalent mesh circuit. The number of impedances in the mesh is $\frac{n}{2}(n - 1)$. In

[1] See the paper "A New Network Theorem," by A. Rosen, *Jour. I.E.E.* (London), Vol. 62, p. 916. In this paper, Mr. Rosen proves the general conversion from a star circuit to a mesh circuit. He derives the necessary formulas and also shows that the mesh, in general, cannot be converted to a star.

terms of impedances, the general conversion formulas from star to mesh are

$$Z_{12} = Z_1Z_2\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}\right) = Z_1Z_2\Sigma\frac{1}{Z} \quad (13)$$

$$Z_{13} = Z_1Z_3\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}\right) = Z_1Z_3\Sigma\frac{1}{Z} \quad (14)$$

$$Z_{mn} = Z_mZ_n\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}\right) = Z_mZ_n\Sigma\frac{1}{Z} \quad (15)$$

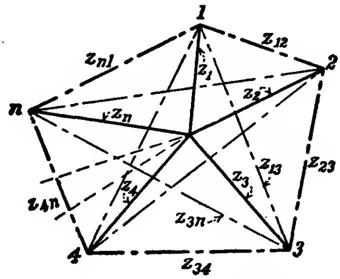

FIG. 3.—Equivalent mesh impedances corresponding to a given star circuit.

In terms of admittances, the corresponding equations become

$$y_{12} = \frac{y_1y_2}{y_1 + y_2 + \cdots + y_n} = \frac{y_1y_2}{\Sigma y} \quad (16)$$

$$y_{13} = \frac{y_1y_3}{y_1 + y_2 + \cdots + y_n} = \frac{y_1y_3}{\Sigma y} \quad (17)$$

$$y_{mn} = \frac{y_my_n}{y_1 + y_2 + \cdots + y_n} = \frac{y_my_n}{\Sigma y} \quad (18)$$

Since these equations are general, they will obviously also hold for the Y-Δ transformation. It is easily shown that, if applied to this case, the general impedance equations reduce to equations (7) to (9), inclusive, and the general admittance equations to equations (10) to (12), inclusive.

EXAMPLE 1

This example illustrates the solution of a simple network problem by the method of simplification.

Statement of Problem

Figure 4 shows a single-wire diagram (simplified) of a railway electrification with trolleys, feeders, and substations.

The assumptions, much simplified, are as follows:

All circuits single phase.

Distribution at 44,000 volts, approximately; trolley at 11,000 volts, approximately; 25 cycles.

Transformers at each substation rated at 8,000 kv.-a., 4 per cent resistance, 7 per cent reactance. Neglect excitation.

Trains, maximum input 6,000 kv.-a. at 80 per cent power factor when trolley voltage is 11,000 volts.

All 44,000-volt transmission lines two No. 00 copper conductors spaced 6 ft.

Trolley No. 000 copper conductor. Rail return. Neglect resistance of rail and ground return. In computing trolley inductance, consider return to be equal diameter conductor spaced 40 ft. Note that substation 3 is fed by a single transmission feeder, while substations 1 and 2 are fed by two identical feeders. These feeders may be considered to be on separate pole lines.

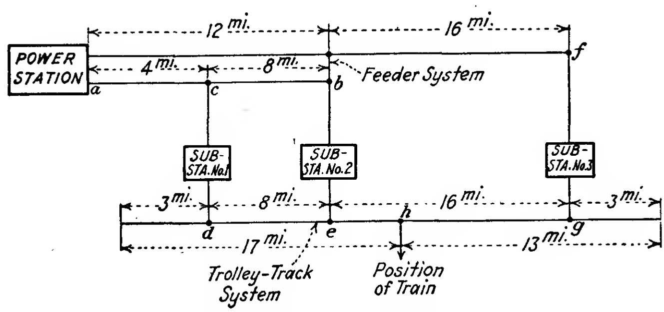

FIG. 4.—Simplified layout of a single-phase railway electrification.

Consider train in the position shown and with controller set so that impedances are those corresponding to maximum input. Compute voltage, current, and power at the train when the bus voltage at the power station is strictly 44,000 volts.

Solution

Circuit Constants:

High-tension Feeders (No. 00 copper):

Resistance[1] = 0.822 ohm/loop-mile

Reactance[1] = 0.630 ohm/loop-mile

Trolley-track System:

Resistance[1] = 0.326 ohm/loop-mile

$$\text{Reactance} = 2\pi f\left(741 \log_{10} \frac{D}{r} + 80.5\right)10^{-6}$$

$$= 2\pi \times 25\left(741 \log_{10} \frac{40 \times 12}{0.2048} + 80.5\right)10^{-6}$$

$$= 0.404 \text{ ohm/loop-mile}$$

[1] Values taken from tables in "Handbook for Electrical Engineers," by H. PENDER, John Wiley and Sons, Inc., New York, 1917.

Substation Transformers:

$$I_{\text{rated}} = \frac{8{,}000 \times 1{,}000}{44{,}000} = 181.8 \text{ amp.}$$

$$\left.\begin{aligned} \text{Resistance} &= \frac{44{,}000 \times 0.04}{181.8} = 9.68 \text{ ohms} \\ \text{Reactance} &= \frac{44{,}000 \times 0.07}{181.8} = 16.92 \text{ ohms} \end{aligned}\right\} \text{Referred to high-tension side}$$

Train:

$$I_{\text{rated}} = \frac{6{,}000 \times 1{,}000}{11{,}000} = 546 \text{ amp.}$$

$$\text{Impedance} = \frac{11{,}000}{546}\underline{/\cos^{-1} 0.8} = 20.14\underline{/36°.9} \text{ ohms}$$

$$\text{Resistance} = 20.14 \times 0.8 = 16.11 \text{ ohms}$$

$$\text{Reactance} = 20.14 \times 0.6 = 12.09 \text{ ohms}$$

Simplification of the Network.—In reducing the system to a single, equivalent impedance between the high-tension bus of the generating station and the train, all impedances will be referred to the high-tension side. Referring to Fig. 4 and to Fig. 5*a*, the impedances of the various sections become

$$\begin{aligned}
Z_{ab} &= (0.822 + j0.630)12 = 9.864 + j7.560 \text{ ohms} \\
Z_{ac} &= (0.822 + j0.630)4 = 3.288 + j2.520 \text{ ohms} \\
Z_{bc} &= (0.822 + j0.630)8 = 6.576 + j5.040 \text{ ohms} \\
Z_{bf} &= (0.822 + j0.630)16 = 13.152 + j10.080 \text{ ohms} \\
Z_{cd} &= Z_{be} = Z_{fg} = 9.68 + j16.992 \text{ ohms} \\
Z_{de} &= (0.326 + j0.404)4^2 \times 8 = 41.76 + j51.68 \text{ ohms} \\
Z_{eh} &= (0.326 + j0.404)4^2 \times 6 = 31.32 + j38.76 \text{ ohms} \\
Z_{gh} &= (0.326 + j0.404)4^2 \times 10 = 52.20 + j64.60 \text{ ohms}
\end{aligned}$$

By adding series impedances wherever possible, the network in Fig. 5*b* is obtained. The two △'s *abc* and *beh* will now be converted to Y's, as indicated by the dotted lines.

Conversion of *abc*

$$\begin{aligned}
Z_{ab} &= 9.864 + j7.560 = 12.44\underline{/37°.5} \\
Z_{ac} &= 3.288 + j2.520 = 4.14\underline{/37°.5} \\
Z_{bc} &= 6.576 + j5.040 = 8.30\underline{/37°.5} \\
\hline
\Sigma Z &= 19.728 + j15.120 = 24.88\underline{/37°.5}
\end{aligned}$$

$$Z_{ai} = \frac{4.14\underline{/37°.5} \times 12.44\underline{/37°.5}}{24.88\underline{/37°.5}} = 2.073\underline{/37°.5} = 1.644 + j1.261$$

$$Z_{bi} = \frac{12.44\underline{/37°.5} \times 8.30\underline{/37°.5}}{24.88\underline{/37°.5}} = 4.15\underline{/37°.5} = 3.288 + j2.522$$

$$Z_{ci} = \frac{4.14\underline{/37°.5} \times 8.30\underline{/37°.5}}{24.88\underline{/37°.5}} = 1.382\underline{/37°.5} = 1.096 + j0.842$$

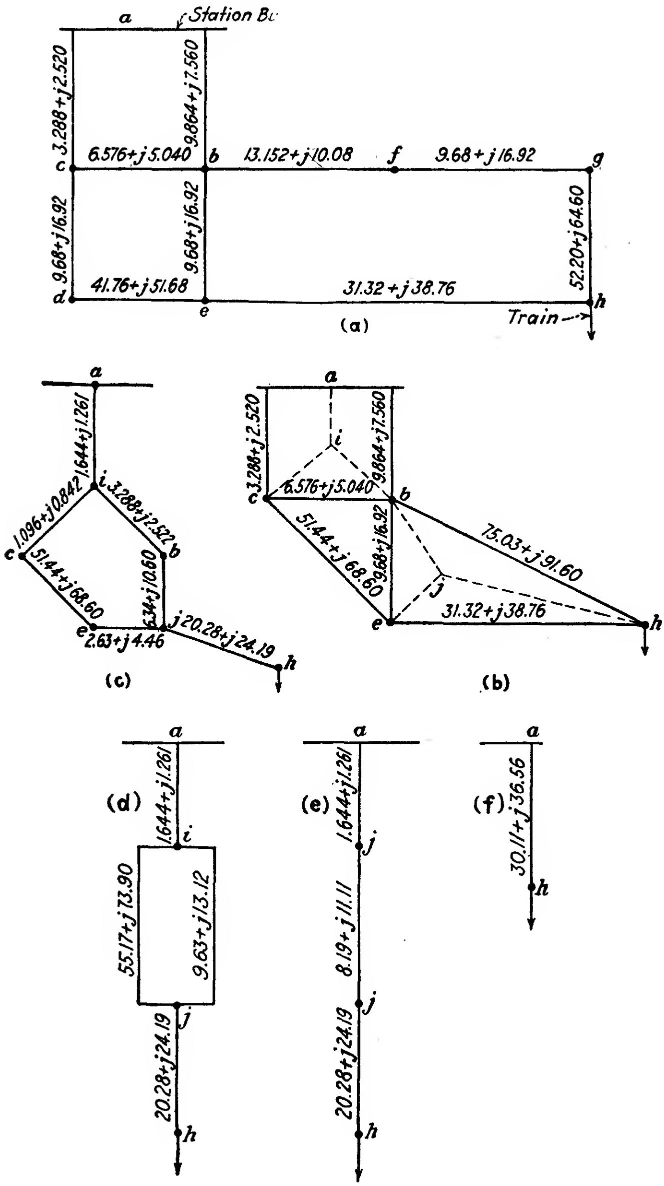

FIG. 5.—Diagrams showing successive steps in the reduction of the circuit in Fig. 4 to a single impedance between the power station and the train.

Conversion of *beh*

$$
\begin{aligned}
Z_{be} &= 9.68 + j16.92 = 19.50\underline{/60^\circ.2} \\
Z_{bh} &= 75.03 + j91.60 = 118.6\underline{/50^\circ.7} \\
Z_{eh} &= 31.32 + j38.76 = 49.77\underline{/51^\circ.1} \\
\hline
\Sigma Z &= 116.03 + j147.28 = 187.2\underline{/51^\circ.8}
\end{aligned}
$$

$$
Z_{bj} = \frac{19.50\underline{/60^\circ.2} \times 118.6\underline{/50^\circ.7}}{187.2\underline{/51^\circ.8}} = 12.35\underline{/59^\circ.1} = 6.34 + j10.60
$$

$$
Z_{hj} = \frac{118.6\underline{/50^\circ.7} \times 49.77\underline{/51^\circ.1}}{187.2\underline{/51^\circ.8}} = 31.55\underline{/50^\circ.0} = 20.28 + j24.19
$$

$$
Z_{ej} = \frac{49.77\underline{/51^\circ.1} \times 19.50\underline{/60^\circ.2}}{187.2\underline{/51^\circ.8}} = 5.18\underline{/59^\circ.5} = 2.63 + j4.46
$$

Using these equivalent Y circuits gives the network (Fig. 5*c*) which immediately reduces to the series-parallel circuit shown in *d*.

$$
\begin{aligned}
Z_{i(ce)j} &= 55.17 + j73.90 = 92.2\underline{/53^\circ.2} \\
Z_{i(b)j} &= 9.63 + j13.12 = 16.27\underline{/53^\circ.7} \\
\hline
\Sigma Z &= 64.80 + j87.02 = 108.7\underline{/53^\circ.3}
\end{aligned}
$$

$$
Z_{ij} = \frac{92.2\underline{/53^\circ.2} \times 16.27\underline{/53^\circ.7}}{108.7\underline{/53^\circ.3}} = 13.80\underline{/53^\circ.6} = 8.19 + j11.11
$$

Substituting this impedance for the two parallel branches gives the circuit (Fig. 5*e*). By addition of the series impedances, the final circuit *f* is obtained. The equivalent impedance between the high-tension bus of the generating station and the train is, hence,

$$Z = 30.11 + j36.56 \text{ ohms}$$

which referred to the low-tension side becomes

$$Z = \frac{30.11 + j36.56}{4^2} = 1.882 + j2.284 \text{ ohms}$$

Current, Voltage, and Power at Train.—The total impedance Z_o referred to the low-tension side becomes

$$
\begin{aligned}
Z_o = Z + Z_t &= 1.882 + j2.284 + 16.11 + j12.09 \\
&= 17.992 + j14.374 = 23.00\underline{/38^\circ.6} \text{ ohms}
\end{aligned}
$$

Train Current:

$$I_t = \frac{V_g}{Z_o} = \frac{11{,}000\underline{/0}}{23.00\underline{/38^\circ.6}} = 477.7\overline{\backslash 38^\circ.6} \text{ amp.}$$

Train Voltage:

$$
\begin{aligned}
V_t = I_t Z_t &= 477.7\overline{/38^\circ.6} \times 20.14\underline{/36^\circ.9} \\
&= 9{,}620\overline{/1^\circ.7} \text{ volts}
\end{aligned}
$$

Train Power:

$$P_t = 477.7^2 \times 16.11 \times 10^{-3} = 3{,}680 \text{ kw.}$$

Short-circuit Current Delivered by a Synchronous Machine.—The currents which flow when a short circuit occurs at a point in a network will depend largely upon the general character of the short-circuit currents in the synchronous machines. When a short circuit is applied, the sustained short-circuit current will, as a rule, not be established immediately. In general, there is a transition period during which the current changes gradually from an initial to a final value (the sustained or steady-state current). The current flowing during the first instants of the transition period may easily be quite considerably greater than the sustained current.

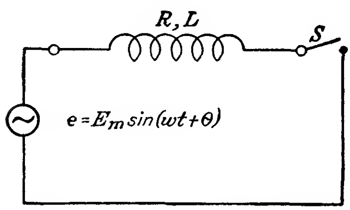

FIG. 6.—A short circuit is suddenly applied to the lumped impedance by closing the switch S.

In order to get an understanding of the character of the short-circuit currents delivered by synchronous machines, it is helpful first to review the case where a short-circuit is suddenly applied to a lumped impedance (Fig. 6) of fixed resistance and inductance at one end of which a constant sinusoidal voltage is maintained. It will be assumed that the reactance part of this impedance is predominant and that the resistance is quite small. This is a condition which would obtain in a synchronous machine.

When a constant sinusoidal voltage is impressed on a constant impedance, it is easy to obtain a rigorous mathematical solution for the current. The differential equation applying to this circuit is readily set up as follows:

$$L\frac{di}{dt} + Ri = E_m \sin(\omega t + \theta) \tag{19}$$

As seen, this is a linear differential equation of the first order with constant coefficients, the solution of which may be expressed as the sum of two terms.[1] Referring to equation (20), which is the solution of equation (19), i_t represents the complementary

$$i = i_t + i_s \tag{20}$$

function and i_s the particular integral. In terms of electrical quantities, the former (i_t) is equal to the transient component of the current, and the latter (i_s) is equal to the steady-state current.[2]

[1] See any standard treatise on differential equations.

[2] The general problem of transients is treated in detail in Vol. II of this treatise.

It is well known that, when the linear differential equation is of the first order, the complementary function consists of a single exponential. The exponent is found by solving the so-called *determinantal equation* obtained by equating the polynomial of derivatives to zero. In this particular case, the determinantal equation becomes

$$Lp + R = 0 \tag{21}$$

where p represents the differential operator. The exponent, hence, becomes

$$p = -\frac{R}{L} \tag{22}$$

The solution for the particular integral, or steady-state current, is readily established by applying the ordinary alternating-current steady-state theory. The complete solution may, hence, be written

$$i = A\epsilon^{-\frac{R}{L}t} + \frac{E_m}{\sqrt{R^2 + (\omega L)^2}} \sin\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right) \tag{23}$$

The constant of integration A is determined by reference to the initial conditions. Assuming zero initial current, this constant becomes

$$A = -\frac{E_m}{\sqrt{R^2 + (\omega L)^2}} \sin\left(\theta - \tan^{-1}\frac{\omega L}{R}\right) \tag{24}$$

and the final solution is given by

$$i = \frac{-E_m \sin\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)}{\sqrt{R^2 + (\omega L)^2}} \epsilon^{-\frac{R}{L}t} + \frac{E_m}{\sqrt{R^2 + (\omega L)^2}} \sin\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right) \tag{25}$$

As seen, the total current consists of a sinusoidally varying component superimposed on a component which decreases exponentially. The amplitude of the steady-state current depends only on the value of the applied voltage and the steady-state impedance of the circuit. The amplitude of the transient, on the other hand, in addition to depending on voltage and impedance, also depends on the point on the voltage wave at which the switch is closed, *i.e.*, it depends on the angle θ. Assuming that the resistance is small, so that the angle of lag of the steady-state current behind the voltage is nearly 90 deg., it is seen that the

transient will entirely disappear when the switch is closed (*i.e.*, short circuit occurs) as the voltage wave passes through one of its maximum values ($\theta = 90$ or 270 deg.). If, on the other hand, the switch is closed on zero voltage ($\theta = 0$ or 180 deg.), the initial amplitude of the transient is equal and opposite to the maximum amplitude of the steady-state current. In this case, therefore,

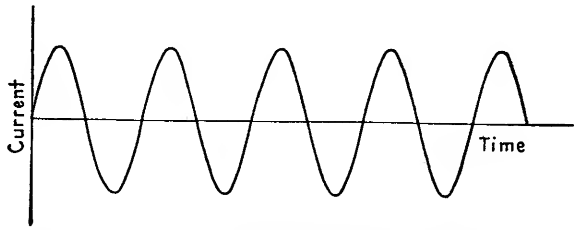

FIG. 7.—The curve shows the steady-state current which will be set up immediately in the circuit in Fig. 6 when the switch is closed as the applied voltage passes through one of its maxima.

the maximum possible total current will be reached after the elapse of a time corresponding to half a cycle.

Figures 7 and 8 illustrate the currents in the two cases. In the former, the steady-state current is set up immediately. In the latter, the wave of total current is a completely "offset" wave obtained by adding the exponential transient and the steady-state

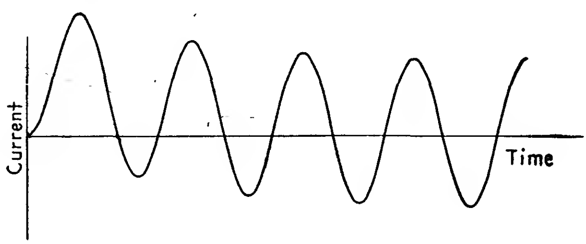

FIG. 8.—The curve shows the current which will flow in the circuit in Fig. 6 when the switch is closed as the applied voltage passes through zero. Mathematically this curve consists of a sinusoid superimposed on an exponential.

sinusoidal current. If the short circuit occurs at values of voltage intermediate between zero and maximum, a "partly offset" curve of total current evidently results.

If the reactance of the synchronous machine[1] were a constant quantity, the character of the transients produced by a short circuit (polyphase or single-phase) applied at the machine

[1] DOHERTY, R. E., and O. E. SHIRLEY, "Reactance of Synchronous Machines and its Applications," *Trans. A.I.E.E.*, p. 1209, 1918.

terminals would be as discussed above, with the exception that, mathematically, the transients would consist of two exponentials instead of merely one. This is due to the inductive coupling between the field and armature circuits. On account of the change in reactance, however, matters become more complicated, and a rigorous mathematical solution is not readily obtained.[1]

When a short circuit is applied at such an instant that the transient disappears, the total current will have the shape illustrated in Fig. 9. As seen, this curve is not offset from the zero axis and may be looked upon as being symmetrical with respect to the latter. It is furthermore seen that the amplitudes decrease as time elapses. This decrease is caused by the fact that the reactance increases from a low initial value to a much higher value

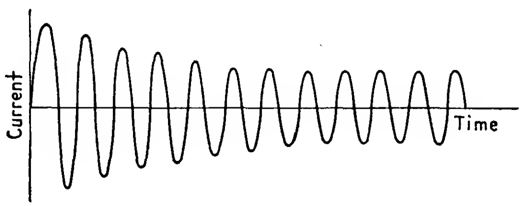

FIG. 9.—Symmetrical short-circuit current delivered by an alternator. Short circuit applied as the voltage wave passes through a maximum value.

for the steady state. Since the flux in the magnetic circuit of the machine cannot change abruptly when the short circuit is applied, the initial current will be limited by the total leakage reactance of the machine. This total leakage reactance is usually called the *transient reactance* and is, in machines without damper windings, equal to the sum of the leakage reactance of the armature and the leakage reactance of the field. If damper windings are present, their effect must be taken into account and the value of the transient reactance modified. Furthermore, eddy currents set up in the field structure will also affect the value of the transient reactance. In general, the armature leakage reactance will constitute the major part of the transient reactance. If the exact value of the transient reactance, therefore, is not known, the armature leakage reactance may be used in place of the transient reactance. As a rule, this will not give rise to appreciable errors

[1] Probably the most up-to-date and rigorous treatment of short-circuit currents in synchronous machines is given in Part IV of a recent paper by R. E. DOHERTY and C. A. NICKLE: "Synchronous Machines, IV. Single-phase Short Circuits," *Jour. A.I.E.E.*, 1928.

except, perhaps, in salient-pole machines without dampers. Furthermore, the discrepancies, if any, will be in a conservative direction, since a reactance which is slightly too low is used.

As time elapses, the armature reaction builds up, and the net flux in the magnetic circuit decreases. The effect of this is equivalent to a gradual increase in the reactance of the machine.

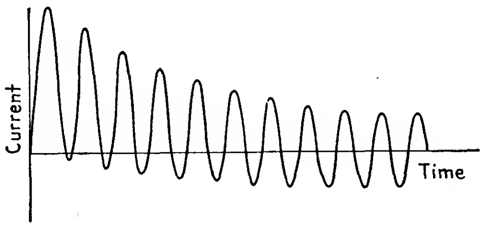

Fig. 10.—Dissymmetrical short-circuit current delivered by an alternator. This curve is completely offset and is obtained when the short circuit is applied as the voltage wave passes through zero.

The reactance finally becomes equal to the synchronous reactance, and steady-state conditions are obtained.

If the short circuit is applied in such a manner that the transient or exponential terms do not disappear, the total current wave will be offset from the horizontal axis and, hence, be dissymmetrical. The total wave may be looked upon as consisting

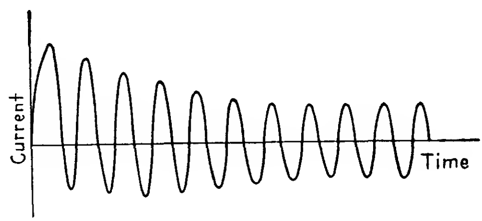

Fig. 11.—Dissymmetrical short-circuit current delivered by an alternator. This curve is not completely offset and is obtained when the short circuit is applied when the voltage has a value between zero and maximum.

of a symmetrical wave of the type shown in Fig. 9 superimposed on the transient components. If the short circuit is applied as the voltage of the phase involved passes through zero, a completely offset wave is obtained, as illustrated in Fig. 10. If the short circuit occurs when the voltage has a value between zero and maximum, a partially offset wave will result, as shown in Fig. 11.

Although the complete story of a short-circuit current is given only when the actual instantaneous values of current are known at various times, it has become standardized practice in practical short-circuit calculations to use effective or root-mean-square values exclusively. The total dissymmetrical wave of current is looked upon as being made up of a direct-current component,

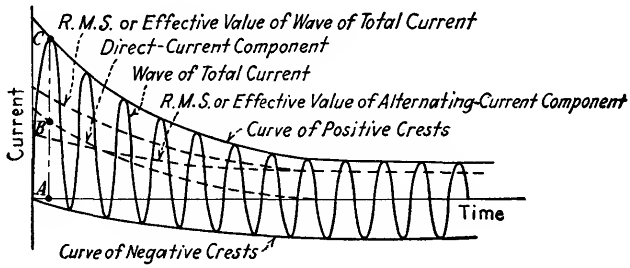

FIG. 12.—This diagram shows how the direct-current and the alternating-current component in a dissymmetrical wave of short-circuit current are determined.

which decreases with time and eventually disappears, and an alternating-current component, the amplitude of which also decreases as time elapses but which finally reaches a steady state. When the total wave of current, as shown, for instance, in Fig. 12, is known, the direct-current component for any value of time may be determined by drawing smooth curves through the maximum positive and negative amplitudes. These curves are

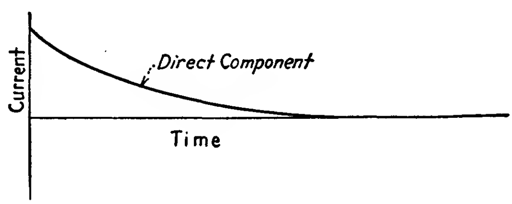

FIG. 13.—Direct-current component in the current wave in Fig. 12.

termed *curves of positive crests* and *negative crests*, respectively. By halving the vertical distance between the curves of positive and negative crests, the amplitude of the direct-current component is obtained, as indicated in Fig. 12, and also shown separately in Fig. 13. Each loop of the total current when considered with respect to the direct-current component is then assumed to represent one half-cycle of a sinusoidal current. Obviously this

is an approximation. The effective value of each of these half-cycles is calculated by dividing the maximum amplitudes with respect to the direct-current component by $\sqrt{2}$. Considering the first maximum in Fig. 12, for instance, the direct-current component is given by

$$I_{\text{d.c.}} = AB \tag{26}$$

and the effective value of the alternating-current component by

$$I_{\text{a.c.}} = \frac{BC}{\sqrt{2}} \tag{27}$$

The alternating-current component plotted alone will give a curve of the type shown in Fig. 9. It is likewise shown in Fig. 14, where a curve of the effective value also has been drawn in. The effective value of the total current is obtained by combining the

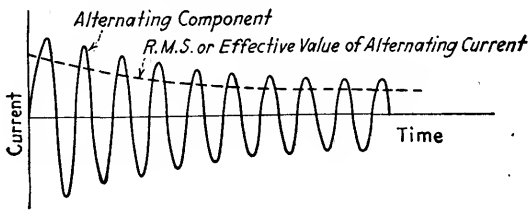

FIG. 14.—Alternating-current component in the wave of short-circuit current in Fig. 12. Both instantaneous and effective values are shown.

direct-current component with the effective value of the alternating-current component for the various values of time. Obviously, the effective value of the total current is equal to the square root of the sum of the squares of the effective values of the components. The combination is, hence, carried out in accordance with the following equation

$$I_{\text{eff.}} = \sqrt{I^2_{\text{d.c.}} + I^2_{\text{a.c.}}} \tag{28}$$

Decrement Factors.—The character of the short-circuit current which will be delivered by a synchronous machine when a short circuit is applied at its terminals has been briefly described above. The short-circuit currents which will flow in a network into which one or more generators feed will be of the same general type. On account of the impedance which the network itself offers, these short-circuit currents will obviously be lower than if the short circuit were applied at the terminals of the machines.

In order to obtain data which might be applicable to general short-circuit calculations, extensive tests have been performed by

the manufacturers, particularly by the General Electric Company and the Westinghouse Electric and Manufacturing Company. These tests comprised the determination of short-circuit currents delivered by machines of various ratings and designs for three general types of short circuit, namely,

1. Symmetrical three-phase short circuit.
2. Single-phase line-to-line short circuit.
3. Line-to-neutral short circuit.

The short circuits were applied directly at the terminals of the machines and also with external reactance between the machines and the short circuit. The external reactance was varied so that a range of total reactance, *i.e.*, external reactance plus machine reactance, between a very low value and 100 per cent, was covered. Reactances up to 15 per cent were inside the machines, and, for higher values, 15 per cent were inside the machines and the remainder external to the machines.

The excitation used in all tests was that corresponding to full load at 80 per cent power factor (lagging). It has been previously

TABLE I.—SYSTEM SHORT-CIRCUIT CURRENT FACTORS APPLICABLE TO THREE-PHASE SHORT CIRCUITS ON THREE-PHASE SYSTEMS

Time from instant of short circuit, seconds	Root-mean-square total current expressed in number of times full-load current for various per cent reactance											
	5 per cent	8 per cent	10 per cent	12 per cent	15 per cent	20 per cent	30 per cent	40 per cent	50 per cent	60 per cent	75 per cent	100 per cent
0.00	35.00	22.00	17.75	14.90	12.00	9.01	6.00	4.52	3.55	2.94	2.36	1.74
0.05	21.18	13.60	11.10	9.40	7.74	5.89	3.98	3.04	2.41	2.03	1.64	1.23
0.08	18.15	11.65	9.50	8.15	6.72	5.14	3.50	2.89	2.15	1.81	1.47	1.11
0.10	16.50	10.70	8.81	7.52	6.22	4.79	3.28	2.54	2.03	1.72	1.40	1.06
0.15	13.48	8.85	7.36	6.32	5.30	4.13	2.87	2.25	1.83	1.56	1.28	0.981
0.20	11.90	7.86	6.56	5.66	4.82	3.74	2.67	2.11	1.72	1.48	1.22	0.943
0.25	10.54	7.10	6.00	5.20	4.45	3.53	2.52	2.01	1.66	1.42	1.18	0.919
0.30	9.56	6.50	5.55	4.85	4.19	3.35	2.42	1.94	1.61	1.39	1.16	0.904
0.40	8.33	5.80	4.96	4.38	3.83	3.10	2.28	1.86	1.55	1.35	1.13	0.888
0.50	7.30	5.15	4.48	3.99	3.52	2.91	2.18	1.79	1.51	1.32	1.11	0.877
0.70	5.94	4.35	3.84	3.48	3.13	2.64	2.04	1.70	1.45	1.27	1.08	0.862
1.00	4.60	3.55	3.24	2.98	2.75	2.38	1.90	1.61	1.39	1.23	1.05	0.843
1.50	3.42	2.90	2.70	2.56	2.43	2.17	1.78	1.54	1.34	1.19	1.03	0.836
2.00	2.72	2.43	2.34	2.27	2.21	2.02	1.71	1.49	1.31	1.17	1.02	0.828
3.00	2.00	2.00	2.00	2.00	2.00	1.88	1.63	1.44	1.28	1.15	1.00	0.820

stated that the magnitude of the short-circuit current depends largely on the value of voltage at the instant the short circuit occurs. In order to be on the conservative side, it is advisable to figure with the maximum possible values. In all these tests, therefore, whether symmetrical three-phase or single-phase, the short circuit was applied in such a manner that maximum possible current was obtained in at least one phase. Oscillograms of this current were taken and analyzed for their root-mean-square values of current in the manner previously described. In this way, it was possible to plot a series of curves of root-mean-square values of short-circuit current *versus* time for different values of reactance.

Different types and sizes of machines would obviously give somewhat different values of short-circuit current. The tests results, however, were compared by the manufacturers responsible for the tests, averaged up, and a set of values selected which was considered representative of the short-circuit currents which machines of average design would deliver. These data in the

TABLE II.—SYSTEM SHORT-CIRCUIT CURRENT FACTORS APPLICABLE TO SINGLE-PHASE LINE-TO-LINE SHORT-CIRCUITS ON THREE-PHASE SYSTEMS

Time from instant of short circuit, seconds	Root-mean-square total current expressed in number of times full-load current for various per cent reactance											
	5 per cent	8 per cent	10 per cent	12 per cent	15 per cent	20 per cent	30 per cent	40 per cent	50 per cent	60 per cent	75 per cent	100 per cent
0.00	35.00	22.02	17.82	14.88	12.00	9.01	6.00	4.52	3.54	2.95	2.36	1.74
0.05	21.80	14.00	11.46	9.69	7.95	6.07	4.12	3.15	2.49	2.08	1.67	1.23
0.08	18.53	12.02	9.90	8.43	7.01	5.38	3.69	2.84	2.25	1.90	1.51	1.11
0.10	16.93	11.10	9.18	7.85	6.56	5.07	3.50	2.70	2.16	1.80	1.44	1.06
0.15	13.92	9.32	7.80	6.75	5.87	4.50	3.16	2.47	1.99	1.67	1.35	0.98
0.20	12.30	8.36	7.09	6.19	5.31	4.21	3.00	2.36	1.90	1.60	1.29	0.94
0.25	11.08	7.66	6.55	5.76	5.00	4.00	2.89	2.29	1.86	1.56	1.26	0.92
0.30	10.18	7.15	6.15	5.45	4.79	3.86	2.82	2.25	1.82	1.54	1.24	0.90
0.40	8.96	6.45	5.62	5.04	4.48	3.67	2.73	2.19	1.79	1.51	1.21	0.89
0.50	8.01	5.89	5.20	4.71	4.24	3.51	2.66	2.15	1.77	1.49	1.20	0.87
0.70	6.73	5.15	4.63	4.27	3.92	3.31	2.57	2.10	1.73	1.47	1.18	0.86
1.00	5.46	4.42	4.08	3.84	3.61	3.12	2.48	2.05	1.70	1.45	1.16	0.84
1.50	4.41	3.81	3.62	3.48	3.35	2.95	2.40	2.01	1.68	1.43	1.15	0.83
2.00	3.69	3.40	3.30	3.23	3.17	2.84	2.35	1.98	1.66	1.41	1.14	0.82
3.00	3.00	3.00	3.00	3.00	3.00	2.73	2.30	1.95	1.64	1.40	1.13	0.81

form of tables and curves[1] are now available for the engineer who has to perform calculations of short-circuit currents in networks.

Instead of giving actual currents in amperes and reactances in ohms, the former are given in terms of rated current of the machine involved and the latter in per cent based on the rating. In this manner, the data are immediately made applicable to all sizes of machines. The factors which indicate the number of times rated current which will flow are called *decrement factors.* Tables I, II, and III give the decrement factors for symmetrical three-phase short circuits, single-phase line-to-line short circuits, and single-phase line-to-neutral short circuits, respectively.

It should be carefully noted that Table III of decrement factors for single-phase line-to-neutral short circuits is of limited appli-

TABLE III.—SYSTEM SHORT-CIRCUIT CURRENT FACTORS APPLICABLE TO SINGLE-PHASE LINE-TO-NEUTRAL SHORT-CIRCUITS ON THREE-PHASE SYSTEMS[1]

Time from instant of short circuit, seconds	Root-mean-square total current expressed in number of times full-load current for various per cent reactance											
	5 per cent	8 per cent	10 per cent	12 per cent	15 per cent	20 per cent	30 per cent	40 per cent	50 per cent	60 per cent	75 per cent	100 per cent
0.00	52.90	33.40	27.10	22.70	19.30	12.80	7.67	5.49	4.05	3.28	2.54	1.85
0.05	33.00	21.25	17.48	14.80	12.80	8.75	5.39	3.91	2.93	2.38	1.85	1.35
0.08	28.50	18.40	15.20	12.89	11.30	7.85	4.85	3.55	2.68	2.20	1.70	1.23
0.10	25.70	16.90	14.05	12.06	10.52	7.38	4.65	3.42	2.58	2.10	1.64	1.19
0.15	21.30	14.40	12.10	10.50	9.35	6.65	4.29	3.20	2.41	1.98	1.52	1.09
0.20	18.80	12.87	10.95	9.60	8.58	6.25	4.10	3.07	2.34	1.91	1.49	1.07
0.25	17.10	11.90	10.10	9.10	8.10	6.00	3.99	3.01	2.30	1.89	1.47	1.05
0.30	15.80	11.10	9.50	8.60	7.80	5.85	3.95	2.96	2.29	1.87	1.46	1.05
0.40	13.90	10.06	8.83	7.96	7.28	5.59	3.86	2.94	2.28	1.86	1.45	1.03
0.50	12.60	9.32	8.20	7.50	6.89	5.46	3.80	2.91	2.26	1.85	1.45	1.02
0.70	10.60	8.19	7.42	6.88	6.46	5.18	3.73	2.87	2.25	1.85	1.44	1.02
1.00	8.71	7.14	6.63	6.22	6.00	4.95	3.66	2.84	2.24	1.84	1.42	1.01
1.50	7.17	6.27	5.98	5.77	5.61	4.76	3.60	2.81	2.23	1.84	1.41	1.01
2.00	6.11	5.67	5.53	5.43	5.35	4.67	3.56	2.80	2.23	1.83	1.40	1.00
3.00	5.10	5.10	5.10	5.10	5.10	4.51	3.52	2.78	2.22	1.83	1.40	1.00

[1] Note the limited applicability of the factors in this table. See statement on p. 22.

[1] Some of the manufacturers' publications contain this information. See also "Relay Handbook," published by the National Electric Light Association, New York, 1926, and the paper "The Application of Decrement Factors in Short-circuit Studies," by W. R. WOODWARD, *Elec. Jour.* p. 213, 1924.

cation in practice. It can be used only with Y-connected generators and then only when (*a*) the neutral points of all generators are solidly grounded and distribution takes place at generator voltage or (*b*) all transformer connections are Y-Y with secondary neutral points solidly grounded and with primary and generator neutral points interconnected. There are but few systems in operation which would fully conform with these specifications as to connections and grounding.

Application of Decrement Factors.—Practical short-circuit calculations making use of the experimentally determined decrement factors involve, as already stated, several assumptions. At this point, these may appropriately be summarized, as follows:

1. Transient characteristics of generators of normal design.
2. The effect of resistance, leakance, and capacitance is neglected.
3. The impedance at the point of short circuit is zero.
4. The excitation of the generators corresponds to full load at 80 per cent power factor (lagging).
5. The short circuit is established at the point of the voltage wave giving maximum possible instantaneous current.
6. There are no voltage regulators.
7. All reactance values up to and including 15 per cent are inside the generators and, for higher values, 15 per cent inside the generators and the remainder external to the generators.

If a single synchronous machine supplies a short circuit through an external reactance and it is desired to determine the amount of current which flows after the elapse of a certain time (t_0), the procedure will obviously be as follows:[1]

The total reactance to the point of short circuit is first determined. This reactance equals the sum of the external reactance and the transient reactance of the machine. In adding these reactances, they are both expressed in per cent, preferably on a base corresponding to the kilovolt-ampere rating of the machine.

The proper table of decrement factors is entered with the percentage total reactance and the time at which the short-circuit current is desired, and the decrement factor selected. The rated current of the machine times the decrement factor is then equal to the root-mean-square value of the short-circuit current which will flow at the elapse of the specified time.

[1] WOODWARD, W. R.: "The Application of Decrement Factors in Short-circuit Studies," *loc. cit.* See also "Relay Handbook," *loc. cit.*

If it is desired to determine the initial symmetrical value of the short-circuit current, this can be obtained by dividing the rated current by the percentage reactance and multiplying by 100. Hence,

$$I_0 = \frac{I_{\text{rated}} \times 100}{X_0} \tag{29}$$

where I_0 represents the initial symmetrical short-circuit current and X_0 the total reactance to the point of short circuit. That the equation is correct may be appreciated from the following consideration: If the reactance were 100 per cent, evidently rated current would flow, assuming that the internal voltage of the machine has its rated value. When the reactance is different from 100 per cent and the internal voltage normal, the ratio of the current which actually flows and the rated current must obviously equal the inverse ratio of the actual reactance and the 100 per cent reactance.

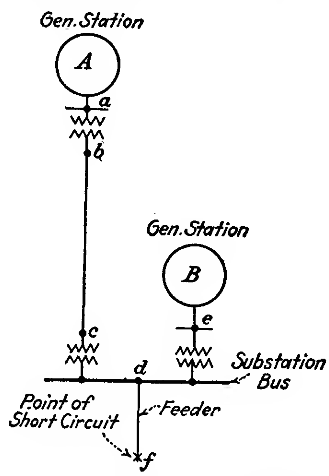

FIG. 15.—Simple two-station system used in discussing application of decrement factors.

When a short circuit occurs on a network, the short-circuit current will, as a rule, be supplied from several generating stations.[1] There are two methods of applying the decrement factors in such cases. Either a single decrement factor may be applied to the synchronous machines lumped, or else a separate decrement factor may be applied to each station. In order to illustrate these methods more fully, the layout shown in Fig. 15 will be considered. This layout involves two generating

[1] If, in one or more of the stations or at other points of the system, there are large synchronous motors or condensers operating, these machines may be treated as generators for the first few seconds after the occurrence of the short circuit. Owing to their inertia, they will continue to run at synchronous speed for a short time and will, during this period, supply current to the short circuit as if they were operating as generators. After a while, these machines will slow down and eventually come to rest. Much before this happens, however, their own circuit breakers may have opened and disconnected them from the system. In no event, therefore, will the synchronous motors or condensers contribute to the sustained short-circuit currents.

stations only but serves to illustrate the principles, since exactly the same methods apply when any number of stations supply the network. In Fig. 15, a short circuit occurs on a feeder a certain distance from the bus of a substation. Generating station A supplies power to the substation over a high-tension transmission line with step-up and step-down transformers. Station B, located near the substation, supplies power to the latter over a bank of step-up transformers.

Figure 16 shows a diagram of the reactances involved. Both stations are connected to a common hypothetical bus at which the normal voltage of the short-circuited feeder is assumed to be maintained. The reactances Aa and Be represent the transient reactances of the two stations. The reactances ab, cd, and ed represent the three banks of transformers; the reactance bc the transmission line; and the reactance df the feeder between the substation bus and the point of short circuit.

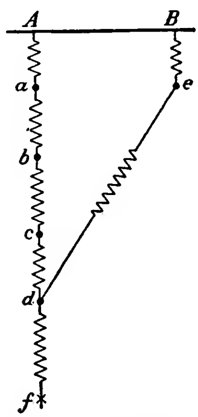

Fig. 16.—Reactance diagram of the system in Fig. 15.

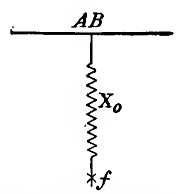

Fig. 17.—Equivalent reactance of the system in Fig. 15.

These reactances should all be expressed in per cent on a common kilovolt-ampere base. The selected kilovolt-ampere base may be taken as the rating of either generating station, as the combined rating of the two stations, or be selected arbitrarily. When the layout involves several generating stations, an arbitrary base of suitable size is most commonly used.

The network reactances are next combined so that the circuit is reduced to a single reactance between the hypothetical bus and the point of short circuit, as shown in Fig. 17. This process of reduction must always be gone through independent of the method of application of decrement factors used.

Method 1. Application of Decrement Factors to the System Lumped.—It will be assumed that the reactances have been cal-

culated on an arbitrary kilovolt-ampere base different from the rating of either station and also different from the combined ratings of the stations. Let the total reactance on this base be X_0.

If a single decrement factor is to be applied to the two stations combined, the per cent total reactance must be converted to a base equal to the combined capacity of the two stations. When the voltage is unchanged, the per cent reactance is directly proportional to the kilovolt-ampere base. Hence, the per cent reactance based on the total rating of the machines is given by

$$X_0' = X_0 \frac{\text{Rating of machines}}{\text{Base kilovolt-ampere}}$$

$$= X_0 \frac{A + B}{\text{Base}} \tag{30}$$

The proper table or curve is now entered with the reactance X_0' and the decrement factor (k) corresponding to the desired time (t_0) selected. The short-circuit current which will flow at the elapse of the time t_0 is then given by

$$I_{\text{s.c.}} = kI_{\text{rated}} \tag{31}$$

Method 2. Application of Decrement Factors to Each Generating Station Separately.—Also, in this case, let the total reactance calculated on some arbitrary base be X_0. In order to determine the proper decrement factor which should be used for each station, it is necessary first to calculate the initial symmetrical current in the short circuit by equation (29). All stations contribute their share to this short-circuit current. The amount which each station supplies is next determined by properly apportioning the total current (I_0) between the stations. In the particular case under discussion, the initial symmetrical short-circuit currents supplied by the stations A and B, respectively, are given by

$$I_{0(A)} = I_0 \frac{X_{Bd}}{X_{Ad} + X_{Bd}} \tag{32}$$

$$I_{0(B)} = I_0 \frac{X_{Ad}}{X_{Ad} + X_{Bd}} \tag{33}$$

Making use of the initial symmetrical short-circuit currents, an equivalent reactance between each station and the point of short circuit is calculated as follows:

$$X_{0(A)} = \frac{I_{\text{rated}(A)} \times 100}{I_{0(A)}} \tag{34}$$

$$X_{0(B)} = \frac{I_{\text{rated}(B)} \times 100}{I_{0(B)}} \tag{35}$$

Since these reactances are computed by means of the rated currents of the machines, they are obtained on the kilovolt-ampere base of the respective machines. The tables or curves of decrement factors may, therefore, be entered with these reactances and a separate decrement factor selected for each station at the desired time. The short-circuit current which each station delivers after the elapse of this time is then obtained by multiplying the rated current of that particular station by its decrement factor. Letting k_A and k_B represent the decrement factors of the two stations, the short-circuit currents at the time t_0 are given by

$$I_{s.c.(A)} = k_A I_{rated(A)} \tag{36}$$

$$I_{s.c.(B)} = k_B I_{rated(B)} \tag{37}$$

The total current flowing in the short circuit is obviously given by the sum of the currents supplied by each machine. Hence,

$$I_{s.c.} = I_{s.c.(A)} + I_{s.c.(B)} \tag{38}$$

In carrying out calculations of this type, it is convenient, particularly when the system involves several generating stations, to arrange the calculations in tabular form. A suitable scheme is suggested in Table IV.

TABLE IV.—SHORT-CIRCUIT CALCULATIONS

Station	Amperes at normal voltage		Per cent equivalent reactance	At time $= t_0$ seconds	
	Rated	Initial symmetrical		Decrement factor	Short-circuit amperes
A............					
B............					

Evidently, of the two methods described above for the application of decrement factors, the first one in which the decrements are applied to the system lumped is the easier one. The second one involves a good deal more labor. The question then arises which one is preferable in a practical case.

Neither method is rigorous even if the general assumptions on which these short-circuit calculations rest are disregarded. When the single decrement factor is applied to all stations combined, the fact that large and small stations, in general, have widely different decrements is not properly taken into account. Hence,

inaccuracies are obviously introduced by lumping the stations. If, on the other hand, a separate factor is applied to each station, the voltage drops in the various branches of the circuit may not balance up at all values of time. Inaccuracies may consequently be anticipated also with this method.

It is difficult to predict, in general, which method will give the better results. It may be said that, as a rule, the first method will give slightly higher values of currents than the second. From this standpoint, the former is preferable in that conservative results are obtained. Although this will be so in most cases, it may not be universally true. In cases where a large station of low reactance is located in the immediate neighborhood of the point of short circuit, currents calculated by the second method may actually be larger than those calculated by the first method, particularly at small values of time. Ordinarily, however, the two methods give results that are very close together. This may be seen, for instance, by comparing the results in Example 2 obtained by computations based on both methods.

Since, as already stated, the first method is the easier and shorter one, it is also the one which is most frequently used in practice.

Solutions by Calculating Table.—In spite of the simplifying assumptions, calculations of short-circuit currents in complicated networks require considerable time and labor. This is obviously a great disadvantage where a frequent check-up on short-circuit currents is necessary, which is the case whenever new extensions and load additions to a system or installations of new circuit breakers and additional equipment are planned.

In order to facilitate determination of short-circuit currents, calculating tables[1] have been used quite extensively. These

[1] Lewis, W. W., "Calculation of Short-circuit Currents in Alternating-current Systems," *Gen. Elec. Rev.*, p. 140, 1919.

Woodward, W. R., R. D. Evans, and C. L. Fortescue, "Calculating Short-circuit Currents in Networks," *loc. cit.* Part I of this article (by Woodward) discusses testing with miniature networks.

Lewis, W. W., "A New Short-circuit Calculating Table," *Gen. Elec. Rev.*, p. 669, 1920.

Corbett, L. J., "A Short-circuit Calculating Table," *Elec. World*, p. 985, 1922.

Dillard, E. W., "A Short-circuit Calculating Table," *Elec. World*, p. 797, 1923.

Schurig, O. R., "Experimental Determination of Short-circuit Currents in Electric Power Networks," *Trans. A.I.E.E.*, p. 10, 1923.

consist of a combination of resistances connected so as to represent the network under consideration. Since in practical short-circuit calculations reactances only are considered, it is possible to represent these on the calculating table by pure resistances and to use direct current. At points where generating stations feed into the actual network, the proper direct-current voltages are impressed on the miniature system. The currents are then simply recorded by direct-current ammeters.

When a calculating table of this type is used, it is obviously possible to obtain values of current for any number of simultaneous short circuits, if this is desired. Ordinarily, however, a short circuit occurs at one point only at a time, so that, as a rule, a single short circuit is all that has to be considered. The currents measured will represent either initial symmetrical short-circuit currents or else sustained short-circuit currents. It all depends upon the values of resistance used to represent the generator reactances. If values corresponding to transient reactances are used, the initial symmetrical short-circuit currents will be measured. If, on the other hand, the generator resistances are adjusted to correspond to synchronous reactances, sustained short-circuit currents will be obtained. It is thus seen that the calculating table does not give the solution at the elapse of a certain time interval. If this is required, decrement factors must be applied, as in the ordinary analytical solution. The equivalent reactance (or reactances), however, with which the decrement tables are entered can be determined by means of the values of current read on the calculating table.

Sometimes the calculating table is permanently set up to represent a specific system. This is the simplest and involves resistances of fixed values only. Many operating companies who use calculating tables for short-circuit determination have tables of this type. As their systems are extended, they simply add the necessary fixed resistances on the calculating table in order to keep the miniature system up to date.

It is obviously also possible to construct calculating tables with a number of variable resistance units. By a convenient plug board or dial arrangement, these resistances may be combined and connected so as to represent any arbitrary system or network within the range of the table. Such calculating tables are flexible and, hence, useful in cases where determination of short-circuit currents is not confined to any specific system. Some of the

manufacturers and consulting-engineering offices have calculating tables of this type.

Recently it has been proposed to extend the use of calculating tables also to the solution of networks under normal operating conditions.[1] In order to make this possible, the miniature system must be built up of impedance units instead of merely resistances, and alternating currents must be used instead of direct currents. The loads in such a system are represented by impedances. The generating stations and other synchronous machines, such as motors and condensers, cannot, in this case, be simulated merely by impressing an alternating voltage of the correct magnitude. It must be ascertained that the voltages also have the correct phase displacement relative to each other. This can be done by using phase shifters[2] to represent the synchronous machines. By these phase shifters, voltages of the correct magnitude as well as phase displacement are obtained.

In addition to being useful for the determination of voltages, currents, and power under normal conditions, the alternating-current calculating table will obviously also give the solution of short-circuit problems. It should actually be superior to the direct-current table for the latter purpose, since it makes it possible also to take resistance into account. Furthermore, its use eliminates the necessity of the assumption that all generator voltages are in phase.

EXAMPLE 2

Statement of Problem

Four generating stations A, B, C, and D feed into a transmission network, as shown in the sketch. The nominal voltages of the various parts of the system, as well as the reactances of the interconnecting lines, are indicated on the diagram, Fig. 18. All reactances given are on a 20,000-kv.-a. base.

Assuming a short circuit on one of the 22-kv. feeders in power station A, it is desired to determine the amount of current that the switch S will have to interrupt after a time of 0.2 sec.

---

[1] Schurig, O. R., "The Solution of Electric Power Transmission Problems in the Laboratory by Miniature Circuits," *Gen. Elec. Rev.*, p. 611, 1923.

———, "A Miniature Alternating-current Transmission System for Network and Transmission-system Problems," *Trans. A.I.E.E.*, p. 831, 1923.

Spencer, H. H., and H. L. Hazen, "Artificial Representation of Power Systems," *Trans. A.I.E.E.* p. 72, 1925.

[2] Described in the paper by Spencer and Hazen, *loc. cit.*

Equipment in Station A:

Two 5,000-kv.-a. generators with 18 per cent reactance.

One 7,812-kv.-a. generator with 25.6 per cent reactance.

One 7,500-kv.-a. synchronous condenser with 85.4 per cent reactance.

The two-circuit transformer in this station has 6.3 per cent reactance, and each of the three-circuit transformers has 5.7 per cent reactance between the 66- and 22-kv. windings; 6.3 per cent reactance between the 66- and 4-kv. windings; and 5.7 per cent reactance between the 22- and 4-kv. windings.

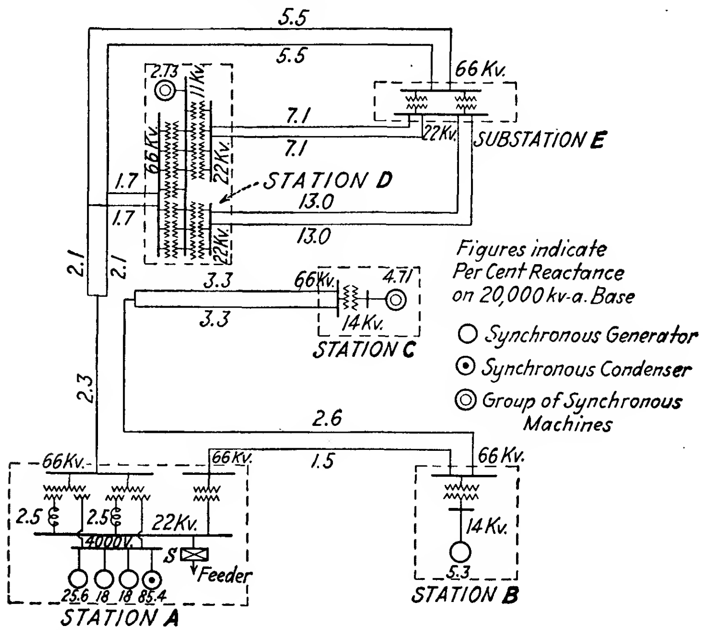

FIG. 18.—Single-line diagram showing layout of power system containing four generating stations and a substation. The short-circuit calculations in Example 2 are based on this layout.

Equipment in Station B:

One 37,500-kv.-a. synchronous generator with 5.3 per cent reactance.

The transformer in this station has 4.4 per cent reactance.

Equipment in Station C:

Total capacity of synchronous machinery 47,050 kv.-a. with reactance of 4.71 per cent.

The transformers in this station have 5.4 per cent reactance.

Equipment in Station D:

Total capacity of synchronous machinery 134,500 kv.-a. with combined reactance of 2.13 per cent.

Four of the seven $^{66}\!/\!_{11}$-kv. transformers have 58.6 per cent reactance each, two have 19.8 per cent reactance each, and one has 12.9 per cent react-

ance. The seven $^{22}\!/_{11}$-kv. transformers are identical, each having 24 per cent reactance.

Equipment in Substation E:

The two $^{66}\!/_{22}$-kv. transformers in this station have 13 per cent reactance each.

The various loads connected to the system have not been indicated, as all load currents are to be neglected as compared to the short-circuit currents. The magnitude and phase of the voltages at the generating stations will be considered the same; in other words, all generators may be assumed connected to one common bus.

Solution

Total reactance of synchronous machines in station A

$$\frac{1}{\frac{1}{25.6}+\frac{2}{18}+\frac{1}{85.4}} = 6.18 \text{ per cent}$$

Each of the three-circuit transformers in this station may be replaced by an equivalent Y-connected network (see Chap. II, equations (70), (71), and (72)). Designating the 66-kv. winding as No. 1, the 22-kv. winding as No. 2, and the 4,000-volt winding as No. 3, the reactances to be assigned to the branches of the equivalent Y become

$$Z_1 = \frac{5.7 + 6.3 - 5.7}{2} = 3.15 \text{ per cent}$$

$$Z_2 = \frac{5.7 + 5.7 - 6.3}{2} = 2.55 \text{ per cent}$$

$$Z_3 = \frac{6.3 + 5.7 - 5.7}{2} = 3.15 \text{ per cent}$$

Total reactance of the seven $^{66}\!/_{11}$-kv. transformers in station D

$$\frac{1}{\frac{4}{58.6}+\frac{2}{19.8}+\frac{1}{12.9}} = 4.05 \text{ per cent}$$

Total reactance of the group of four $^{22}\!/_{11}$-kv. transformers in station D

$$^{24}\!/_{4} = 6.0 \text{ per cent}$$

Total reactance of the group of three $^{22}\!/_{11}$-kv. transformers in station D

$$^{24}\!/_{3} = 8.0 \text{ per cent}$$

Using the values calculated above for machine and transformer combinations in conjunction with the rest of the data given on the layout (Fig. 18), the circuit diagram (Fig. 19*a*) results. By going through the steps indicated in Fig. 19*a* to *g*, the circuit is reduced to a single reactance between a hypothetical 22-kv. bus and the point of short circuit. This reactance is 4.63 per cent.

$$\text{Base current} = \frac{20{,}000}{\sqrt{3} \times 22} = 525 \text{ amp.}$$

Initial symmetrical short-circuit current corresponding to 4.63 per cent reactance

$$I_0 = \frac{525 \times 100}{4.63} = 11{,}340 \text{ amp.}$$

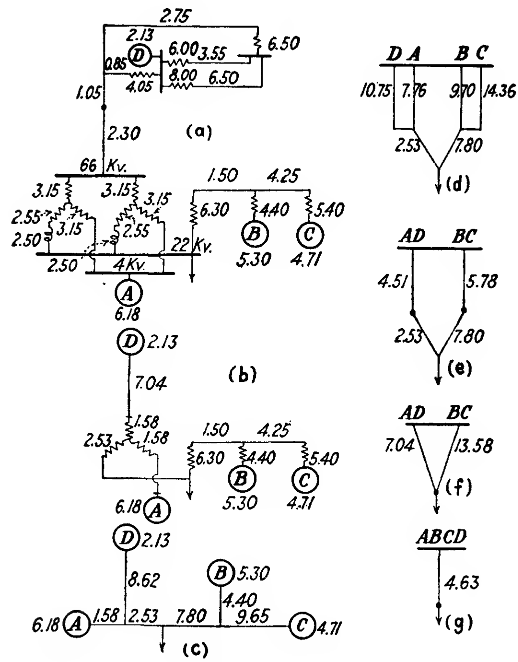

FIG. 19.—Diagrams showing successive steps in reducing the system in Fig. 18 to a single equivalent reactance between the hypothetical bus and the point of short circuit.

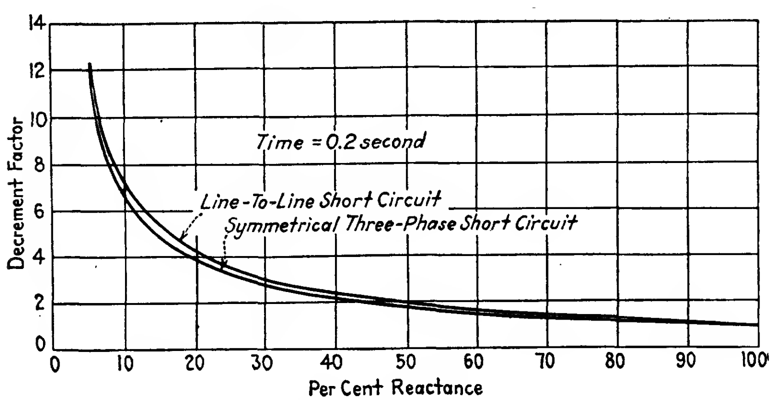

FIG. 20.—Decrement factors for symmetrical three-phase short circuits and single-phase line-to-line short circuits plotted *versus* reactance for a definite time $t_0 = 0.2$ sec. Data for these curves have been obtained from Tables I and II.

1. *Application of Decrements to Each Generating Station Separately.*—The initial symmetrical short-circuit current divides among the four generating stations A, B, C, and D as follows:

$$\text{From } A \text{ and } D \quad \frac{11{,}340 \times 4.63}{7.04} = 7{,}460 \text{ amp.}$$
$$\text{From } B \text{ and } C \quad = 3{,}880 \text{ amp.}$$
$$\text{From } A \quad \frac{7{,}460 \times 4.51}{7.76} = 4{,}340 \text{ amp.}$$
$$\text{From } D \quad = 3{,}120 \text{ amp.}$$
$$\text{From } B \quad \frac{3{,}880 \times 5.78}{9.70} = 2{,}310 \text{ amp.}$$
$$\text{From } C \quad = 1{,}570 \text{ amp.}$$

Obtaining the decrement factors from the curves (Fig. 20) plotted from data in tables I and II, the short-circuit currents supplied by each station at the end of 0.2 sec. are calculated in the table below.

Stations	Amperes at 22 kv.		Per cent equivalent reactance	Three-phase short circuit		Line-to-line short circuit	
	Rated	Calcu-lated		Factor at 0.2 sec.	Amperes at 0.2 sec.	Factor at 0.2 sec.	Amperes at 0.2 sec.
A........	664	4,340	15.30	4.7	3,120	5.2	3,450
B........	985	2,310	42.6	2	1,970	2.2	2,170
C........	1,236	1,570	78.7	1.15	1,420	1.24	1,530
D........	3,530	3,120	113	0.82	2,900	0.82	2,900
Total....	.....	.....	......	....	9,410	....	10,050

The circuit breaker in the 22-kv. feeder will, hence, have to interrupt after 0.2 sec.

Three-phase short circuit.................... 9,400 amp.
Line-to-line short circuit..................... 10,000 amp.

2. *Application of Decrements to the Generating Stations Lumped.*—Total rated current capacity of the four stations A, B, C, and D combined

$$I_{\text{rated}} = 664 + 985 + 1{,}236 + 3{,}530 = 6{,}415 \text{ amp.}$$

$$\text{Equivalent reactance} = \frac{6{,}415 \times 100}{11{,}340} = 56.6 \text{ per cent}$$

From the decrement curves (Fig. 20) are obtained the following decrement factors at $t = 0.2$ sec.:

Three-phase short circuit........................... 1.55
Line-to-line short circuit............................ 1.69

The circuit breaker in the feeder will interrupt after 0.2 sec.:

Three-phase short circuit.................. $6{,}415 \times 1.55 = 9{,}940$ amp.
Line-to-line short circuit.................... $6{,}415 \times 1.69 = 10{,}830$ amp.

CHAPTER II

TRANSFORMER IMPEDANCE AND EQUIVALENT CIRCUITS

General Theory of Multicircuit Transformers.—Consider a transformer having its coils connected in such a manner that there exist n independent circuits. In this n-circuit transformer, each circuit has a definite resistance, self-inductance, and mutual inductance with respect to every one of the other circuits. Employing the classical equations for coupled circuits, the instantaneous terminal voltage of each winding can readily be expressed in terms of these constants and the instantaneous currents in the various windings. The following equations result:

$$v_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + M_{12}\frac{di_2}{dt} + M_{13}\frac{di_3}{dt} + \cdots + M_{1n}\frac{di_n}{dt} \qquad (1)$$

$$v_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + M_{21}\frac{di_1}{dt} + M_{23}\frac{di_3}{dt} + \cdots + M_{2n}\frac{di_n}{dt} \qquad (2)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$v_n = R_n i_n + L_n \frac{di_n}{dt} + M_{n1}\frac{di_1}{dt} + M_{n2}\frac{di_2}{dt} + M_{n3}\frac{di_3}{dt} + \cdots \qquad (3)$$

In these equations, R and L with appropriate subscripts represent the resistance and self-inductance of the various circuits and M the mutual inductance between the two windings designated by the double subscript attached. It should be noted that the order of these subscripts is insignificant, *viz.*, $M_{1n} = M_{n1}$, etc.

If the n-circuit transformer considered is an air-core transformer, the self- and mutual inductances are strictly constant. If the transformer has an iron core, as is always the case in commercial practice, the self- and mutual inductances are variables, being functions of the saturation and, hence, of the instantaneous currents. The difficulty caused by the non-constancy of these fundamental parameters, which is inherent with iron-core transformers, may, in a great many problems, be removed by the following device:

The magnetic fluxes which give rise to the flux linkages corresponding to the self- and mutual inductances may be divided into two components, one which is confined to the iron core and another which wholly or partly exists in air. The former is by far the greater portion of the total flux, but the latter and smaller part determines almost entirely the operating characteristics of the transformer. It will now be assumed that the flux confined to the iron core depends only upon the *value* of the magnetomotive force producing it and is entirely unaffected by the *position* of this magnetomotive force with respect to the core. While this is not precisely true, the error should be very small indeed. In other words, the iron flux is assumed to be the same whether produced by a given number of ampere turns in circuit one, two, three or n.

Let the flux in the iron contribute the part M_c of the mutual inductances. Also, assume for simplicity that all windings have the same number of turns or that all constants involved are reduced to the same base (referred to the same winding) in the well-known manner by multiplying resistances and self-inductances by the squares of the proper ratios of turns and mutual inductances by the direct ratios of turns. Equations (1), (2), and (3) may then be rewritten as follows:

$$v_1 = R_1 i_1 + (L_1 - M_c)\frac{di_1}{dt} + (M_{12} - M_c)\frac{di_2}{dt} + (M_{13} - M_c)\frac{di_3}{dt} +$$
$$\cdots + (M_{1n} - M_c)\frac{di_n}{dt} + M_c\frac{d}{dt}(i_1 + i_2 + i_3 + \cdots + i_n) \quad (4)$$

$$v_2 = R_2 i_2 + (L_2 - M_c)\frac{di_2}{dt} + (M_{21} - M_c)\frac{di_1}{dt} + (M_{23} - M_c)\frac{di_3}{dt} +$$
$$\cdots + (M_{2n} - M_c)\frac{di_n}{dt} + M_c\frac{d}{dt}(i_1 + i_2 + i_3 + \cdots + i_n) \quad (5)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$v_n = R_n i_n + (L_n - M_c)\frac{di_n}{dt} + (M_{n1} - M_c)\frac{di_1}{dt} + (M_{n2} - M_c)\frac{di_2}{dt} +$$
$$(M_{n3} - M_c)\frac{di_3}{dt} + \cdots + M_c\frac{d}{dt}(i_1 + i_2 + i_3 + \cdots + i_n) \quad (6)$$

In these equations, the inductances corresponding to the differences $(L_1 - M_c)$, $(M_{12} - M_c)$, etc., are *sensibly constant quantities*, since they are due to fluxes which fully or partly exist in air. The last term in each equation represents the voltage

induced in each winding by the flux exclusively confined to the iron core.

Considering voltages and currents of a *single frequency only*, equations (4) to (6) inclusive may be rewritten in vector form.

$$
\begin{aligned}
V_1 = R_1I_1 + j\omega(L_1 - M_c)I_1 + j\omega(M_{12} - M_c)I_2 \\
+ j\omega(M_{13} - M_c)I_3 + \cdots + j\omega(M_{1n} - M_c)I_n \\
+ j\omega M_c(I_1 + I_2 + I_3 + \cdots + I_n) \quad (7)
\end{aligned}
$$

$$
\begin{aligned}
V_2 = R_2I_2 + j\omega(L_2 - M_c)I_2 + j\omega(M_{21} - M_c)I_1 \\
+ j\omega(M_{23} - M_c)I_3 + \cdots + j\omega(M_{2n} - M_c)I_n \\
+ j\omega M_c(I_1 + I_2 + I_3 + \cdots + I_n) \quad (8)
\end{aligned}
$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$
\begin{aligned}
V_n = R_nI_n + j\omega(L_n - M_c)I_n + j\omega(M_{n1} - M_c)I_1 \\
+ j\omega(M_{n2} - M_c)I_2 + j\omega(M_{n3} - M_c)I_3 + \cdots \\
+ j\omega M_c(I_1 + I_2 + I_3 + \cdots + I_n) \quad (9)
\end{aligned}
$$

Introducing, in general,

$$X_{nn} = \omega(L_n - M_c) \quad (10)$$

$$X_{n1} = X_{1n} = \omega(M_{n1} - M_c) = \omega(M_{1n} - M_c) \quad (11)$$

$$X_{n2} = X_{2n} = \omega(M_{n2} - M_c) = \omega(M_{2n} - M_c) \quad (12)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and

$$E_c = j\omega M_c(I_1 + I_2 + I_3 + \cdots + I_n) \quad (13)$$

the equations for the terminal voltages reduce to

$$V_1 = (R_1 + jX_{11})I_1 + jX_{12}I_2 + jX_{13}I_3 + \cdots + jX_{1n}I_n + E_c \quad (14)$$

$$V_2 = (R_2 + jX_{22})I_2 + jX_{21}I_1 + jX_{23}I_3 + \cdots + jX_{2n}I_n + E_c \quad (15)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$V_n = (R_n + jX_{nn})I_n + jX_{n1}I_1 + jX_{n2}I_2 + jX_{n3}I_3 + \cdots + E_c \quad (16)$$

Here, X_{11} is the *self leakage reactance* of circuit 1, and X_{12} is the *mutual leakage reactance* between circuits 1 and 2. These reactances are due to fluxes which wholly or partly exist in air. Hence they are *very nearly* constant and independent of saturation. The significance of the other reactances is at once apparent.

By subtracting each of the above equations from the preceding one, E_c is eliminated and a new set obtained giving the *difference*

between the terminal voltages or the effective impedance drop of pairs of windings, as follows:

$$V_1 - V_2 = [R_1 + j(X_{11} - X_{21})]I_1 - [R_2 + j(X_{22} - X_{12})]I_2 + j(X_{13} - X_{23})I_3 + \cdots + j(X_{1n} - X_{2n})I_n \quad (17)$$

$$V_2 - V_3 = [R_2 + j(X_{22} - X_{32})]I_2 - [R_3 + j(X_{33} - X_{23})]I_3 + j(X_{21} - X_{31})I_1 + \cdots + j(X_{2n} - X_{3n})I_n \quad (18)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$V_n - V_1 = [R_n + j(X_{nn} - X_{1n})]I_n - [R_1 + j(X_{11} - X_{n1})]I_1 + j(X_{n2} - X_{12})I_2 + j(X_{n3} - X_{13})I_3 + \cdots \quad (19)$$

The difference between the self-reactance of a winding and the mutual reactance between this and one of the other windings is the *true leakage reactance* of the first winding with respect to the other. Thus, $(X_{11} - X_{21})$ is the true leakage reactance of winding 1 with respect to winding 2. Similarly, $(X_{22} - X_{12})$ is the true leakage reactance of winding 2 with respect to winding 1, and, in general, $(X_{nn} - X_{mn})$ is the true leakage reactance of winding n with respect to winding m. The relative aspect of the leakage reactances should be carefully noted. The leakage reactance of a winding is not a quantity which is dependent upon and characteristic of that winding alone; it must, of necessity, be defined with respect to some other winding. In an n-winding transformer, therefore, the true leakage reactance of one of the windings may, in general, have $n - 1$ values, namely, a distinct value for each of the other windings with respect to which the leakage reactance is determined. Of course, there is a possibility that two or more of these values may coincide, due, for instance, to symmetrical arrangement of the windings.

It is convenient to introduce symbols for the true leakage impedances and reactances as follows:

$$Z_{1(2)} = R_1 + jX_{1(2)} = R_1 + j(X_{11} - X_{21}) \quad (20)$$

$$Z_{2(1)} = R_2 + jX_{2(1)} = R_2 + j(X_{22} - X_{12}) \quad (21)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Z_{n(m)} = R_n + jX_{n(m)} = R_n + j(X_{nn} - X_{mn}) \quad (22)$$

$Z_{n(m)}$, for instance, represents the leakage impedance of winding n with respect to winding m. Hence, the first subscript refers to the winding itself, and the second one in parenthesis indicates the winding with respect to which the leakage reactance is considered.

Substituting these abbreviations, equations (17) to (19) inclusive become

$$V_1 - V_2 = Z_{1(2)}I_1 - Z_{2(1)}I_2 + j(X_{13} - X_{23})I_3 + \cdots + j(X_{1n} - X_{2n})I_n \quad (23)$$

$$V_2 - V_3 = Z_{2(3)}I_2 - Z_{3(2)}I_3 + j(X_{21} - X_{31})I_1 + \cdots + j(X_{2n} - X_{3n})I_n \quad (24)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$V_n - V_1 = Z_{n(1)}I_n - Z_{1(n)}I_1 + j(X_{n2} - X_{12})I_2 + j(X_{n3} - X_{13})I_3 + \cdots \quad (25)$$

The sum of the currents in the several windings is equal to the exciting current. Hence, the following relation holds:

$$I_1 + I_2 + I_3 + \cdots + I_n = I_e \quad (26)$$

In many practical problems, however, the exciting current is ignored, since it is small compared with the load currents. On this basis, the sum of the currents is zero, *i.e.*,

$$I_1 + I_2 + I_3 + \cdots + I_n = 0 \quad (27)$$

The voltage equations previously given (equations (14) to (16) or (23) to (25) inclusive) used in conjunction with one of the current equations (equations (26) or (27)) suffice for the solution of any single-phase or polyphase transformer problem. Of course, the constants involved, as well as a sufficient number of terminal conditions, must be known.

In the following, the general theory will be specifically applied to two-circuit, three-circuit, and four-circuit transformers.

Two-circuit Transformers. *Application of the General Equations.*—In the case of a two-circuit transformer, the general equations (14) to (16) inclusive reduce to

$$V_1 = (R_1 + jX_{11})I_1 + jX_{12}I_2 + E_c \quad (28)$$

$$V_2 = (R_2 + jX_{22})I_2 + jX_{21}I_1 + E_c \quad (29)$$

The difference between the terminal voltages or the effective impedance drop becomes

$$V_1 - V_2 = [R_1 + j(X_{11} - X_{21})]I_1 - [R_2 + j(X_{22} - X_{12})]I_2 = Z_{1(2)}I_1 - Z_{2(1)}I_2 \quad (30)$$

Here, $Z_{1(2)}$ and $Z_{2(1)}$ are the true leakage impedances of windings 1 and 2, respectively, with respect to the other winding. Since, in a two-winding transformer, however, there can never be any doubt about the proper relative aspect of these impedances, the subscripts in parentheses may be omitted, thus simplifying the notation. With multicircuit transformers, on the other

hand, the double subscripts must be retained. Equation (30) may, hence, be written

$$V_1 - V_2 = Z_1 I_1 - Z_2 I_2 \tag{31}$$

If excitation is taken into consideration, the sum of the primary and secondary currents equals the exciting current

$$I_1 + I_2 = I_e \tag{32}$$

Combining equations (31) and (32) gives

$$\begin{aligned} V_1 - V_2 &= (Z_1 + Z_2)I_1 - Z_2 I_e \\ &= -(Z_1 + Z_2)I_2 + Z_1 I_e \end{aligned} \tag{33}$$

In practical calculations, the exciting current is frequently neglected. This is permissible in very many problems, since the exciting current seldom exceeds 5 per cent of the full-load current and, hence, exerts but a small influence on the actual values of current, power, losses, and efficiency. Ignoring excitation, equation (33) reduces to

$$\begin{aligned} V_1 - V_2 &= (Z_1 + Z_2)I_1 = -(Z_1 + Z_2)I_2 \\ &= Z_{12} I_1 = -Z_{12} I_2 \end{aligned} \tag{34}$$

Here, Z_{12}, equal to the sum of the separate leakage impedances, represents the equivalent impedance of the transformer. This quantity is ordinarily determined by the standard short-circuit test and is the only constant required when the excitation is not taken into account. The composition of the equivalent impedance is at once apparent from the following equation:

$$Z_{12} = R_e + jX_e = R_1 + R_2 + j(X_1 + X_2) \tag{35}$$

Equivalent Network of Two-circuit Transformers.—Equations (31) and (32) indicate that a T-circuit, as shown in Fig. 21, is the logical equivalent network of a two-circuit transformer. The separate leakage impedances Z_1 and Z_2 make up the arms of the T, while the impedance of the pillar Z_c carrying the exciting current is given by

$$Z_c = \frac{V_1 - I_1 Z_1}{I_e} \tag{36}$$

In general, the separate leakage impedances of the two windings are not equal and the equivalent T-circuit representing the transformer will consequently be dissymmetrical. Very often, however, it is assumed that the equivalent impedance splits equally between the two windings. This assumption is fre-

quently necessary, since enough data for a correct determination of the separate impedances are seldom at hand. Fortunately, the symmetrical T-circuit thus obtained is sufficiently accurate for most practical problems, with the exception of such as specifically deal with the distribution of the harmonic components of the exciting currents between primary and secondary circuits.

Making use of equation (28), equation (36) may also be written

$$Z_c = \frac{E_c + jX_{12}I_e}{I_e} = \frac{E_c}{I_e} + jX_{12} \tag{37}$$

which again may be modified to

$$Z_c = \frac{j\omega M_c I_e}{I_e} + j\omega(M_{12} - M_c) = j\omega M_{12} \tag{38}$$

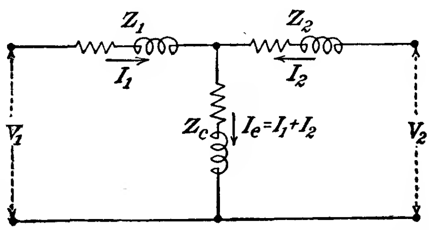

FIG. 21.—Equivalent network of a two-circuit transformer.

Equation (38) shows that, basing the constants of the equivalent circuit on the general theory, the pillar impedance Z_c contains reactance only. Furthermore, the value of this reactance corresponds to the mutual inductance between the two windings. It will be constant, therefore, only when the mutual inductance is a constant quantity or, what amounts to the same thing, where the magnetization curve is a straight line. Hence, the representation will be exact for air-core transformers. For iron-core transformers, the equivalent circuit is exact only at the particular value of saturation (voltage) at which the impedance Z_c is determined. At other values of saturation (voltage), the representation is, of necessity, more or less approximate, depending upon the shape of the magnetization curve and the position of the operating point on the latter.

Although it might be desirable from the standpoint of accuracy to use several values for Z_c when the equivalent circuit is used for the determination of performance at widely different voltages, this is usually not done. The reason is, of course, that the effect of the exciting current is fundamentally small. It is customary,

therefore, to determine the pillar impedance only at a saturation corresponding to normal voltage.

As already mentioned, equation (38) shows the impedance Z_c to contain nothing but reactance. This is due to the fact that the classical theory of coupled circuits fails to consider the effect of hysteresis and eddy currents in the iron. The equivalent circuit, therefore, derived directly from this theory, makes no provision for the proper consideration of the core loss. This loss may be accounted for by assigning also a resistance to the pillar impedance. This resistance is given such a value that the fictitious copper loss developed in it by the exciting current is equal to the core loss. Obviously, exact correspondence is also here obtained only at some particular value of voltage, since the relation between core loss and saturation is not exactly quadratic.

In practice, the pillar impedance, or excitation impedance (admittance), as it is commonly termed, is determined by an open-circuit test; *i.e.*, one winding is excited while the other is left open and, hence, carries no current. For this condition, equation (36) becomes

$$Z_c = \frac{V_1}{I_e} - Z_1 = (R_1 + j\omega L_1) - Z_1 \tag{39}$$

The leakage impedance Z_1 is negligible compared to the impedance $R_1 + j\omega L_1$ corresponding to the self-inductance. To substantiate this, consider a transformer having 10 per cent equivalent impedance and 5 per cent exciting current. The sum of the excitation impedance and the primary leakage impedance from equation (39) corresponding to 5 per cent exciting current is 2,000 per cent. Assuming that the primary leakage impedance equals one-half the equivalent impedance, the actual value of the excitation impedance becomes 1,995 per cent. Obviously, it is immaterial whether this impedance is considered to be 1,995 or 2,000 per cent, the difference between the two values being but one-quarter of 1 per cent. Hence, the excitation impedance is found by simply dividing the normal impressed voltage by the exciting current, giving

$$Z_c = R_c + jX_c = \frac{V_1}{I_e} \tag{40}$$

and

$$Y_c = \frac{1}{R_c + jX_c} = \frac{I_e}{V_1} \tag{41}$$

Since a T-circuit is always convertible into an equivalent Π-circuit, the transformer may evidently also be represented by the latter type of network. This is seldom done, however, since, in general, the T representation is more convenient.

When the consideration of excitation is omitted, the transformer circuit reduces to a single impedance. This impedance is equal to the equivalent impedance Z_{12} of the transformer, as indicated by equation (34).

There are but few problems involving transformers which actually *require* that the exciting current be taken into account. This is true when the analysis is concerned with the conditions in the power circuit alone. Where inductive interference is involved, the exciting currents, and particularly their higher harmonic components, may be just the currents that should be considered.

As examples of power problems where the effect of the excitation should be included may be mentioned unbalanced operation of the Y-Y-connected transformer bank without primary neutral and long-distance transmission where transformer banks are connected to long lines of considerable capacitance. The former case is rather unimportant, since very few Y-Y connections of this type exist. The latter, however, is exceedingly important. The lagging exciting current of the transformers tends to reduce the effect of the leading charging current of the line, and, if neglected, an entirely erroneous picture may be had of the conditions at the transformer terminals.

Three-circuit Transformers.[1] *Application of the General Equations.*—In the case of a three-circuit transformer, the general equations (14) to (16) inclusive reduce to

$$V_1 = (R_1 + jX_{11})I_1 + jX_{12}I_2 + jX_{13}I_3 + E_c \tag{42}$$
$$V_2 = (R_2 + jX_{22})I_2 + jX_{21}I_1 + jX_{23}I_3 + E_c \tag{43}$$
$$V_3 = (R_3 + jX_{33})I_3 + jX_{31}I_1 + jX_{32}I_2 + E_c \tag{44}$$

By taking differences between pairs of the above equations, the effective impedance drops in circuits 1 and 2; circuits 2 and 3 and circuits 3 and 1, respectively, become

[1] Boyajian, A., "Theory of Three-circuit Transformers." *Trans. A.I.E.E.*, p. 508, 1924.

Peters, J. F., "Three-winding Transformers," *Elec. Jour.*, pp. 12 and 71, 1925.

Discussion by W. V. Lyon in *Trans. A.I.E.E.*, p. 813, 1925.

$$V_1 - V_2 = [R_1 + j(X_{11} - X_{21})]I_1 - [R_2 + j(X_{22} - X_{12})]I_2 + j(X_{13} - X_{23})I_3 \quad (45)$$

$$V_2 - V_3 = [R_2 + j(X_{22} - X_{32})]I_2 - [R_3 + j(X_{33} - X_{23})]I_3 + j(X_{12} - X_{13})I_1 \quad (46)$$

$$V_3 - V_1 = [R_3 + j(X_{33} - X_{13})]I_3 - [R_1 + j(X_{11} - X_{31})]I_1 + j(X_{23} - X_{12})I_2 \quad (47)$$

The sum of the currents in the three circuits equals the exciting current, *viz.*

$$I_1 + I_2 + I_3 = I_e \quad (48)$$

Introducing this relation in equations (45), (46), and (47), these may be written

$$V_1 - V_2 = [R_1 + j(X_{11} - X_{21} + X_{23} - X_{13})]I_1 - [R_2 + j(X_{22} - X_{12} + X_{13} - X_{23})]I_2 + j(X_{13} - X_{23})I_e \quad (49)$$

$$V_2 - V_3 = [R_2 + j(X_{22} - X_{32} + X_{13} - X_{12})]I_2 - [R_3 + j(X_{33} - X_{23} + X_{12} - X_{13})]I_3 + j(X_{12} - X_{13})I_e \quad (50)$$

$$V_3 - V_1 = [R_3 + j(X_{33} - X_{13} + X_{12} - X_{23})]I_3 - [R_1 + j(X_{11} - X_{31} + X_{23} - X_{12})]I_1 + j(X_{23} - X_{12})I_e \quad (51)$$

The symbols Z_1, Z_2, and Z_3 will now be introduced to represent the relatively complex impedances in the brackets above, as follows:

$$Z_1 = R_1 + j(X_{11} - X_{12} + X_{23} - X_{13}) \quad (52)$$

$$Z_2 = R_2 + j(X_{22} - X_{32} + X_{13} - X_{12}) \quad (53)$$

$$Z_3 = R_3 + j(X_{33} - X_{13} + X_{12} - X_{23}) \quad (54)$$

It is interesting to note the composition of the reactance part of these impedances. Consider Z_1, for instance. Here $X_{11} - X_{12}$ is the leakage reactance of winding 1 with respect to winding 2. X_{23} and X_{13} are the mutual reactances between windings 2 and 3 and between windings 1 and 3, respectively. Hence, the effective or composite reactance associated with circuit 1 is the leakage reactance of circuit 1 with respect to circuit 2 plus the differential effect of circuit 3 upon circuits 1 and 2. If the third circuit were symmetrically located with respect to the other two circuits, its mutual effect upon each would be the same. X_{23} would equal X_{13}, and the effective reactance or circuit 1 would be its leakage reactance with respect to circuit 2 alone. Or, if the third circuit did not carry any current, the mutual reactances X_{23} and X_{13} would not enter into the effective reactance of the first circuit at all. The problem then immediately reduces to a two-circuit transformer problem.

Another interesting feature is the possibility of negative effective reactance. Again referring to the impedance Z_1, it is conceivable that the circuits might be so arranged that the mutual reactances between circuits 1 and 2 and between circuits 1 and 3 are relatively large in comparison with the mutual reactance between circuits 2 and 3. If the former are sufficiently predominant in magnitude, the effective reactance of Z_1 may become negative and have the effect of a capacitive reactance.

The difference between the mutual reactances appearing in connection with the exciting current in equations (49) to (51) inclusive may be written as follows:

$$j(X_{13} - X_{23}) = Z_2 - Z_{2(1)} = Z_{1(2)} - Z_1 \tag{55}$$
$$j(X_{12} - X_{13}) = Z_3 - Z_{3(2)} = Z_{2(3)} - Z_2 \tag{56}$$
$$j(X_{23} - X_{12}) = Z_1 - Z_{1(3)} = Z_{3(1)} - Z_3 \tag{57}$$

Hence, the mutual reactance differences in question can always be determined as the difference between the effective "three-circuit" impedance of one of the windings and the leakage impedance of the same winding with respect to one of the others.

Equations (49), (50), and (51) may now be written in the following simplified form:

$$V_1 - V_2 = Z_1I_1 - Z_2I_2 + (Z_2 - Z_{2(1)})I_e \tag{58}$$
$$V_2 - V_3 = Z_2I_2 - Z_3I_3 + (Z_3 - Z_{3(2)})I_e \tag{59}$$
$$V_3 - V_1 = Z_3I_3 - Z_1I_1 + (Z_1 - Z_{1(3)})I_e \tag{60}$$

These equations properly applied will solve any three-circuit transformer problem where a sufficient number of terminal conditions is known. They provide for taking the exciting current into account, if this refinement is desired. Although the exciting current varies slightly with the load, it would usually be considered constant and be given the value which corresponds to normal saturation. It would, hence, be taken equal to the current flowing when rated voltage is impressed on one of the windings, the other two being open.

The exciting current, however, is very seldom included in the calculations. As a rule, it is neglected. When this is the case, the currents in the three circuits add to zero, and equations (58) to (60) inclusive reduce to

$$V_1 - V_2 = Z_1I_1 - Z_2I_2 \tag{61}$$
$$V_2 - V_3 = Z_2I_2 - Z_3I_3 \tag{62}$$
$$V_3 - V_1 = Z_3I_3 - Z_1I_1 \tag{63}$$

Determination of Impedances.—The effective impedances (equations (52), (53), and (54)) to be assigned to the three windings are readily determined by three short-circuit tests. They are, hence, obtained at practically zero saturation. The equivalent impedance of pairs of windings are measured exactly as for the two-circuit transformer. Since the exciting current for this condition is entirely negligible, the currents in the two windings under test are equal and opposite. Assuming that power is supplied to circuit 1 with circuit 2 short-circuited, then to circuit 2 with circuit 3 short-circuited, and, finally, to circuit 3 with circuit 1 short-circuited, equations (61), (62), and (63) give

$$V_1 = (Z_1 + Z_2)I_1 = Z_{12}I_1 \tag{64}$$
$$V_2 = (Z_2 + Z_3)I_2 = Z_{23}I_2 \tag{65}$$
$$V_3 = (Z_3 + Z_1)I_3 = Z_{31}I_3 \tag{66}$$

The equivalent impedances Z_{12}, Z_{23}, and Z_{31} are thus obtained. The effective impedances of the three circuits are then calculated from

$$Z_1 + Z_2 = Z_{12} \tag{67}$$
$$Z_2 + Z_3 = Z_{23} \tag{68}$$
$$Z_3 + Z_1 = Z_{31} \tag{69}$$

Simultaneous solution of these equations gives

$$Z_1 = \frac{Z_{12} + Z_{31} - Z_{23}}{2} \tag{70}$$
$$Z_2 = \frac{Z_{23} + Z_{12} - Z_{31}}{2} \tag{71}$$
$$Z_3 = \frac{Z_{31} + Z_{23} - Z_{12}}{2} \tag{72}$$

Equivalent Network of Three-circuit Transformers.—It is impracticable to represent the three-circuit transformer by an exact equivalent network when the exciting current is taken into account. Inspection of equations (58) to (60) inclusive will show the futility of attempting such representation.

Not so, however, when the effect of excitation is omitted. Equations (61) to (63) inclusive show that, in this case, the three-circuit transformer may be represented by an equivalent Y-connected network, as indicated in Fig. 22. This representation is simple and very convenient in many instances.

In three-phase connections, the individual transformers may also be represented by Y-connected circuits of the type shown in

Fig. 22. A complete three-phase arrangement is indicated in Fig. 23. It should be noted, however, that this representation is not practical except when conditions are perfectly balanced so that the three-phase problem reduces to a single-phase problem. When unbalance is involved, the solution must be based on the equations themselves rather than on an equivalent network.

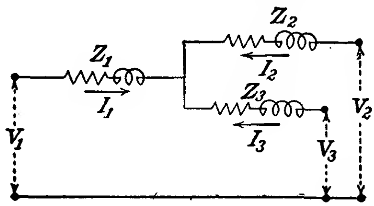

FIG. 22.—Equivalent network of a three-circuit transformer. Excitation neglected.

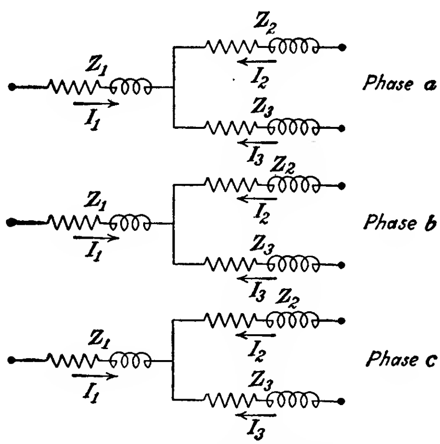

FIG. 23.—Three-phase arrangement of equivalent networks of three-circuit transformers. Excitation neglected.

Since a Y-connected circuit, in general, is convertible to a Δ-connected circuit, it is evident that the latter may also be used to represent a three-circuit transformer. As a rule, however, the equivalent Y-connected network is more convenient from the standpoint of calculation.

EXAMPLE 1

Statement of Problem

A 2,300-volt generating station supplies power to two short transmission lines through a bank of three 2,100 kv.-a. three-circuit transformers. The

transmission lines are both three-phase, and their voltage ratings are 110 and 22 kv., respectively. The transformers are connected Δ-Δ-Δ.

The nominal voltages of each single-phase transformer are as follows:

Winding 1..........	2,300 volts
Winding 2..........	22,000 volts
Winding 3..........	110,000 volts

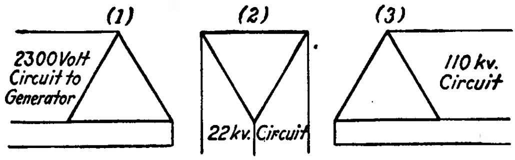

FIG. 24.—Three-circuit transformer layout for Example 1.

The equivalent short-circuit reactances are

6.10 per cent between the 2.3- and the 22-kv. windings
6.59 per cent between the 2.3- and the 110-kv. windings
6.15 per cent between the 22- and the 110-kv. windings

If the 110-kv. load at the transformer terminals is 3,150 kv.-a. at 85 per cent power factor (lagging), the 22-kv. load at the transformer terminals 3,150 kv.-a. at 90 per cent power factor (lagging), and the high-tension voltage strictly 110 kv., calculate

1. The voltage on the 22-kv. circuit.
2. The voltage of the generator.
3. The voltage regulation when the 110-kv. load is disconnected.

Solution

Ratios of Transformation:

$$\frac{\text{Winding 2}}{\text{Winding 1}} = \frac{22}{2.3} = 9.56$$

$$\frac{\text{Winding 3}}{\text{Winding 1}} = \frac{110}{2.3} = 47.8$$

$$\frac{\text{Winding 3}}{\text{Winding 2}} = \frac{110}{22} = 5$$

The circuit layout is shown in Fig. 24, and the equivalent Y-connected, single-phase network in Fig. 25. The reactances of the latter are (equations (70), (71), and (72))

$$X_1 = \frac{6.10 + 6.59 - 6.15}{2} = 3.27 \text{ per cent}$$

$$X_2 = \frac{6.15 + 6.10 - 6.59}{2} = 2.83 \text{ per cent}$$

$$X_3 = \frac{6.59 + 6.15 - 6.10}{2} = 3.32 \text{ per cent}$$

1. Consider the nominal voltages and the base currents calculated from the full rating at the nominal voltages to be the 100 per cent values of voltages and currents, respectively.

The voltage which would appear at the junction point of the equivalent circuit is given by

$$V_m = V_3 - jI_3X_3 \tag{a}$$

$$I_3 = 50 \text{ per cent of base current}$$

$$\cos \phi_3 = 0.85 \quad \sin \phi_3 = 0.527$$

Using V_3 as standard phase,

$$V_m = 100 + 0.5(0.85 - j0.527)j3.32$$
$$= 100.88 + j1.41 = 100.9 \text{ per cent}$$

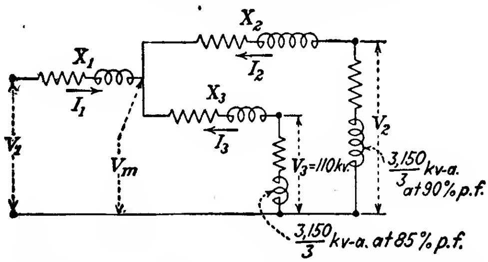

FIG. 25.—Equivalent network of one of the three-circuit transformers in Fig. 24 with loads attached.

This voltage, however, is also given by

$$V_m = V_2 - jI_2X_2 \tag{b}$$

$$I_2 = \frac{50 \times 100}{V_2} \text{ per cent of base current}$$

$$\cos \phi_2 = 0.90 \quad \sin \phi_2 = 0.436$$

Using V_2 as standard phase,

$$100.9\underline{/\alpha} = V_2 + \frac{50}{V_2}(0.9 - j0.436)j2.83 = V_2 + \frac{61.1}{V_2} + j\frac{127.2}{V_2}$$

Squaring gives

$$100.9^2 = V_2^2 + 122.2 + \frac{61.1^2}{V_2^2} + \frac{127.2^2}{V_2^2}$$

which again reduces to

$$V_2^4 - 10{,}050\, V_2^2 + 20{,}008 = 0$$

The solution of this equation is

$$V_2 = \sqrt{5{,}025 \pm \sqrt{25.23 \times 10^6 - 0.02 \times 10^6}}$$
$$= \sqrt{5{,}025 \pm 5{,}021} = 100.2 \text{ per cent}$$

The voltage on the 22-kv. circuit is, therefore,

$$22 \times 1.002 = 22.04 \text{ kv.}$$

2. As seen, the phase displacement between the voltage at the junction point V_m and the terminal voltage V_3 is very small, being less than 0.8 deg. The displacement between V_m and V_2 is still smaller. It is sufficiently accurate, therefore, particularly since the resistances have been neglected, to consider the phase angle of the currents with respect to the junction-point voltage as being the same as with respect to the terminal voltages.

$$I_2 = \frac{50 \times 100}{100.2} = 49.9 \text{ per cent}$$

$$I_1 = -(I_2 + I_3) \tag{c}$$

$$I_1 = 49.9(0.9 - j0.436) + 50(0.85 - j0.527)$$
$$= 86.4 - j48.1$$

The generator voltage is given by

$$V_1 = V_m + jI_1X_1 \tag{d}$$

Using V_3 as standard phase,

$$\begin{aligned} V_1 &= 100.88 + j1.41 + (0.864 - j0.481)j3.27 \\ &= 102.45 + j4.23 = 102.5 \text{ per cent} \end{aligned}$$

The voltage at generator is, hence,

$$2{,}300 \times 1.025 = 2{,}360 \text{ volts}$$

3. In calculating the regulation, it will be assumed that the size and power factor of the 22-kv. load remains the same in spite of the change in voltage.

With no load on the 110-kv. circuit the following equation holds:

$$V_1 = V_2 + I_1Z_{12} = V_2 - I_2Z_{12} \tag{e}$$

Using V_2 as standard phase

$$102.5\underline{/\beta} = V_2 + \frac{50}{V_2}(0.9 - j0.436)j6.10 = V_2 + \frac{133}{V_2} + j\frac{274.3}{V_2}$$

Hence,

$$102.5^2 = V_2^2 + 266 + \frac{133^2}{V_2^2} + \frac{274.3^2}{V_2^2}$$

which reduces to

$$V_2^4 - 10{,}230\, V_2^2 + 93{,}100 = 0$$

The solution of this is

$$\begin{aligned} V_2 &= \sqrt{5{,}115 \pm \sqrt{26.19 \times 10^6 - 0.09 \times 10^6}} \\ &= \sqrt{5{,}115 \pm 5{,}108} = 101.1 \text{ per cent} \end{aligned}$$

The regulation of the 22-kv. circuit is, therefore,

$$\frac{(101.1 - 100.2)100}{100.2} = 0.9 \text{ per cent}$$

The voltage of the 110-kv. windings is given by

$$V_3 = V_1 - I_1Z_1 \tag{f}$$

$$I_1 = I_2 = \frac{50 \times 100}{101.1} = 49.4 \text{ per cent}$$

Again neglecting the small displacement between the voltages V_1 and V_2 as far as the phase of the current is concerned and considering V_1 as standard phase, equation (*f*) gives

$$\begin{aligned} V_3 &= 102.5 - 0.494(0.9 - j0.436)j3.27 \\ &= 101.8 - j1.45 = 101.9 \text{ per cent} \end{aligned}$$

The regulation of the 110-kv. circuit is, hence,

$$101.9 - 100 = 1.9 \text{ per cent}$$

Four-circuit Transformers. *Application of the General Equations.*—In the case of a four-circuit transformer, the general equations (14) to (16) reduce to[1]

$$V_1 = I_1(R_1 + jX_{11}) + jX_{12}I_2 + jX_{13}I_3 + jX_{14}I_4 + E_c \tag{73}$$
$$V_2 = I_2(R_2 + jX_{22}) + jX_{12}I_1 + jX_{23}I_3 + jX_{24}I_4 + E_c \tag{74}$$
$$V_3 = I_3(R_3 + jX_{33}) + jX_{13}I_1 + jX_{23}I_2 + jX_{34}I_4 + E_c \tag{75}$$
$$V_4 = I_4(R_4 + jX_{44}) + jX_{14}I_1 + jX_{24}I_2 + jX_{34}I_3 + E_c \tag{76}$$

Also, neglecting excitation,

$$I_1 + I_2 + I_3 + I_4 = 0 \tag{77}$$

[1] See discussion by W. V. Lyon in *Trans. A.I.E.E.*, p. 816, 1925.

These five fundamental equations may be handled in a variety of ways. Since there is some advantage in having them in symmetrical form, the following method of elimination will be used: First eliminate E_c by taking successive differences. Then eliminate I_4 from the first difference, I_1 from the second difference, and I_2 from the third difference. This gives

$$\begin{aligned}
V_1 - V_2 = {} & I_1[R_1 + j(X_{11} - X_{12} - X_{14} + X_{24})] - \\
& I_2[R_2 + j(X_{22} - X_{12} + X_{14} - X_{24})] + \\
& \qquad j(X_{13} - X_{23} + X_{24} - X_{14})I_3 \qquad (78) \\
V_2 - V_3 = {} & I_2[R_2 + j(X_{22} - X_{23} - X_{12} + X_{13})] - \\
& I_3[R_3 + j(X_{33} - X_{23} + X_{12} - X_{13})] + \\
& \qquad j(X_{13} - X_{34} + X_{24} - X_{12})I_4 \qquad (79) \\
V_3 - V_4 = {} & I_3[R_3 + j(X_{33} - X_{34} - X_{23} + X_{24})] - \\
& I_4[R_4 + j(X_{44} - X_{34} + X_{23} - X_{24})] + \\
& \qquad j(X_{24} - X_{14} + X_{13} - X_{23})I_1 \qquad (80)
\end{aligned}$$

These three equations together with equation (77) are sufficient to determine the currents, in any case. An examination of these equations shows some interesting facts in regard to the impedances. In the first equation, $R_1 + j(X_{11} - X_{12} - X_{14} + X_{24})$ is the impedance that would be assigned to the first winding if the first, second, and fourth windings were considered as a three-circuit transformer. This impedance will be represented by Z_{124}. The impedance $R_2 + j(X_{22} - X_{12} + X_{14} - X_{24})$ is that which would be assigned to the second winding if the first, second, and fourth were considered as a three-circuit transformer. This will be represented by Z_{214}. The first subscript shows to which winding the impedance is attached. The second and third subscripts indicate which of the other windings are grouped with the first to form a three-circuit transformer. The order of the second and third subscripts is unimportant; that is, there is no difference between Z_{124} and Z_{142}. It will also be noticed that the coefficient of I_3 in equation (78) is $Z_{213} - Z_{214}$, that the coefficient of I_4 in equation (79) is $Z_{342} - Z_{312}$, and that the coefficient of I_1 in equation (80) is $Z_{413} - Z_{423}$. Thus, the equations of voltage differences may be written as follows:

$$V_1 - V_2 = I_1 Z_{124} - I_2 Z_{214} + I_3(Z_{213} - Z_{214}) \qquad (81)$$
$$V_2 - V_3 = I_2 Z_{231} - I_3 Z_{321} + I_4(Z_{342} - Z_{312}) \qquad (82)$$
$$V_3 - V_4 = I_3 Z_{342} - I_4 Z_{432} + I_1(Z_{413} - Z_{423}) \qquad (83)$$

There are some other interesting relations between these impedances. For example,

$$Z_{124} - Z_{123} = -Z_{214} + Z_{213} \tag{84}$$
$$Z_{134} - Z_{132} = -Z_{314} + Z_{312} \tag{85}$$
$$Z_{421} - Z_{423} = -Z_{241} + Z_{243} \tag{86}$$

Determination of Impedances.—Since the impedances involved in the solution of four-circuit transformer problems are the same as in the three-circuit case, they can be determined in a similar manner. Having obtained by test the equivalent impedances of pairs of windings, the desired effective impedances are calculated by

$$Z_{124} = \frac{Z_{12} + Z_{41} - Z_{24}}{2} \tag{87}$$

$$Z_{214} = \frac{Z_{24} + Z_{12} - Z_{41}}{2} \tag{88}$$

$$Z_{213} = \frac{Z_{23} + Z_{12} - Z_{31}}{2} \tag{89}$$

$$Z_{321} = \frac{Z_{31} + Z_{23} - Z_{12}}{2} \tag{90}$$

$$Z_{342} = \frac{Z_{34} + Z_{23} - Z_{24}}{2} \tag{91}$$

$$Z_{432} = \frac{Z_{24} + Z_{34} - Z_{23}}{2} \tag{92}$$

$$Z_{413} = \frac{Z_{34} + Z_{41} - Z_{31}}{2} \tag{93}$$

Very little, if anything, is gained by attempting to represent the four-circuit transformer by an equivalent network.

Determination of Separate Leakage Reactances.—The separate leakage reactances of transformer windings cannot, in general, be calculated with accuracy. The standard formulas found in textbooks on principles and design of transformers are all based on broad assumptions in regard to the distribution of the leakage flux and may easily give results which are in error to a considerable extent. Furthermore, it seems to be doubtful whether more rigorous and reliable formulas are capable of being developed.

For precise determination of these reactances, therefore, experimental methods must be resorted to. Theoretically, there are several tests for this purpose, some single-phase[1] and some

[1] BOYAJIAN, A., "Resolution of Transformer Reactance into Primary and Secondary Reactances," *Trans. A.I.E.E.*, p. 805, 1925.

three-phase.[1] Some of these tests give the leakage reactances at operating flux density (or any desired value of flux density, for that matter), while some determine the leakage reactances at a very low (practically zero) saturation. The latter corresponds closely to the conditions under which the equivalent impedance is obtained by a short-circuit test. Since there undoubtedly is *some* change in the leakage reactances when there is a material change in saturation, these various tests may not give identically the same results.

When the saturation is decreased from that corresponding to normal operation to approximately zero, the value of the equivalent reactance will increase, due to the reduction in reluctance of the iron paths of the leakage fluxes. It is believed that the increase in equivalent reactance will not exceed 10 per cent and usually will be less than this figure. The separate leakage reactances, therefore, will also increase but not necessarily in the same proportion, since, as a rule, a large drop in saturation will be accompanied by a slight redistribution of leakage reactance between the windings. In a core-type transformer with cylindrical coils, for instance, a change in saturation affects the leakage reactance of the winding nearest to the core to a larger extent than the leakage reactance of the winding farther away from the core. This is exactly what might be expected, since the leakage flux of the former has a relatively longer path in iron than the leakage flux of the latter.

It should be noted that leakage-reactance changes of the order of magnitude mentioned above manifest themselves only when rather large changes occur in the flux densities. In particular, some change in the leakage reactances may be expected when one value of the flux density corresponds to a condition where the iron is more or less saturated, while, at the other value, the effect of saturation is absent; in other words, where the two operating points in question lie on *each* side of the bend of the magnetization curve. It should also be noted, however, that even for quite considerable changes of flux density in the operating region, *i.e.*, *above* the knee of the magnetization curve, *the leakage reactance remains sensibly constant.*[2]

[1] Dahl, O. G. C., "Separate Leakage Reactance of Transformer Windings," *Trans. A.I.E.E.*, p. 785, 1925.

[2] Substantiation of this statement will be found in the experiments described in the paper "Separate Leakage Reactance of Transformer Windings," by O. G. C. Dahl, *loc. cit.*

Evidently, it is desirable to ascertain the values of the leakage reactances at as nearly operating density as possible. Hence, from this standpoint, the tests which determine them at a high density are preferable. All of the tests, however, are not likely to give the same inherent precision. In calculating the leakage impedances from test data, *voltages and currents of a single frequency must be used.* When the waves are distorted, oscillograms must be taken and the components of the desired frequency singled out by analysis. In some of the tests, it is theoretically immaterial which one of the harmonics is used. Making use of the component which is the largest percentage of the composite wave, however, will give the maximum accuracy. In some tests, the largest component may be the fundamental; in others, the third harmonic. From the standpoint of precision of measurement, therefore, the best tests are either those which involve metering of pure waves of either fundamental or higher-harmonic frequency or those where any distortion likely to occur will be so small that the principal component of the wave is capable of exact determination by oscillogram analysis.

Single-phase Tests.—There are two single-phase tests which will give the actual values of the separate leakage impedances and one which will give their ratio. The latter, therefore, will suffice only when the sum of the leakage impedances or the equivalent impedance is known. The single-phase tests are applicable to both two-circuit and multicircuit transformers and may be used to give the leakage impedance of one winding with respect to any other winding. The determination of the separate leakage impedances is fundamentally a two-circuit problem, even in a multi-circuit transformer and, in outlining the tests below, two-circuit transformers will be assumed. If it is desired to determine all the possible separate leakage impedances involved in a multicircuit transformer, it merely means repetition of the same tests with appropriate changes of connections, so that each winding is considered with respect to every one of the others.

It is assumed that the transformers considered have unity ratio of transformation. If the ratio is different from unity, potential transformers must be used in the test involving measurement of voltage drop due to exciting current and in the parallel-conjunction test. A current transformer must be used in the series-opposition test. These auxiliary transformers should have the same ratio of transformation as the main transformer under test.

The auxiliary transformers may introduce errors due to incorrect ratio and also due to the phase displacement between their primary and secondary voltages and currents.

a. Drop Due to Exciting Current. Excitation of One Winding Only.—In this test, one of the windings is excited, as shown in Fig. 26, and the exciting current measured. One terminal of the other winding is connected to the excited winding in such a manner that the difference between the terminal volt-

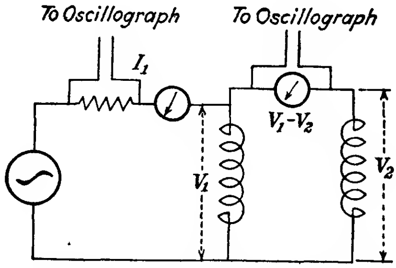

FIG. 26.—Single-phase determination of separate leakage impedances by measuring voltage drop due to exciting current.

ages of the two windings can be measured as indicated with a voltmeter drawing negligible current. Since the secondary winding carries no current, the voltage difference thus measured is directly equal to the primary leakage impedance drop due to the exciting current. The general equations are

$$V_1 = (R_1 + jX_{11})I_1 + jX_{12}I_2 + E_c \tag{94}$$
$$V_2 = (R_2 + jX_{22})I_2 + jX_{21}I_1 + E_c \tag{95}$$

For $I_2 = 0$, subtraction of these equations gives

$$V_1 - V_2 = [R_1 + j(X_{11} - X_{21})]I_1 = Z_1I_1 \tag{96}$$

Hence,

$$Z_1 = \frac{V_1 - V_2}{I_1} \tag{97}$$

As seen, the leakage impedance of winding 1 with respect to winding 2 is immediately obtained. By repeating the test with winding 2 as primary and winding 1 as secondary, the leakage impedance of winding 2 with respect to winding 1 is also readily determined.

Both the measured current and the voltage drop will contain harmonics. Since the equations are applicable to quantities of a single frequency only, oscillograms must be taken and the desired harmonic components separated out by analysis. Figure 27

shows an oscillogram taken during a test of this type. The fundamental component of the exciting current will always be the largest. Usually, also, the fundamental component of the voltage drop will be larger than any of the harmonics, except, perhaps, at very high saturations where the third-harmonic component may be the greatest. As a rule, therefore, maximum precision should be obtained by using the fundamental compo-

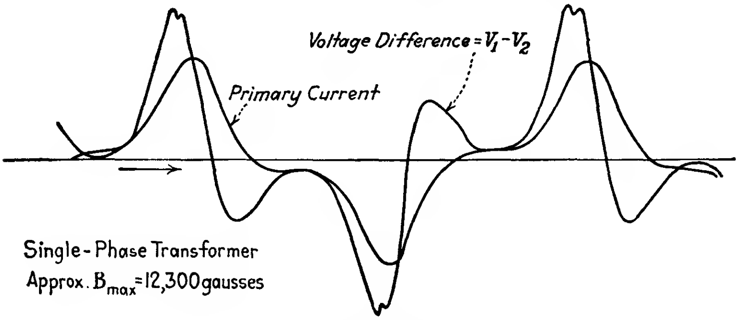

FIG. 27.—Oscillogram from leakage-impedance test using the circuit connections shown in Fig. 26.

nents of voltage drop and current in calculating the leakage impedance.

Even though the fundamental components be used, however, the precision of this test leaves something to be desired. This is due, of course, to the inherent difficulty of analyzing a complex wave with great accuracy. Only when the component which is wanted is predominant can *exact* determination be expected.

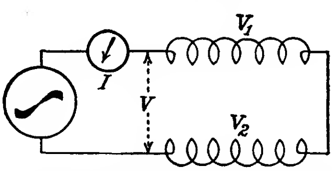

FIG. 28.—Single-phase determination of separate leakage impedances by the series-opposition test. Oscillographic measurements are not necessary in this case, as both voltages and currents are sinusoidal.

b. Series-opposition Test.—If the two windings are connected in series opposition and an alternating-current voltage impressed, as shown in Fig. 28, no flux will exclusively exist in the core. This is readily seen from the general equations which, with the windings in opposition, may be written

$$V_1 = (R_1 + jX_{11})I_1 - jX_{12}I_2 + j\omega M_c(I_1 - I_2) \qquad (98)$$

$$V_2 = (R_2 + jX_{22})I_2 - jX_{12}I_1 - j\omega M_c(I_1 - I_2) \qquad (99)$$

Since the two windings carry the same current, *i.e.*, since

$$I_1 = I_2 = I \tag{100}$$

the last term in equations (98) and (99) is zero, which means that there is no net magnetization of the core as a whole. These equations, therefore, reduce to

$$V_1 = (R_1 + jX_1)I = Z_1 I \tag{101}$$

$$V_2 = (R_2 + jX_2)I = Z_2 I \tag{102}$$

which show that the terminal voltage of each winding is directly equal to its separate leakage-impedance drop. The total voltage impressed on the two windings in series evidently equals the equivalent leakage-impedance drop. Thus,

$$V = V_1 + V_2 = (Z_1 + Z_2)I = Z_{12}I \quad . \tag{103}$$

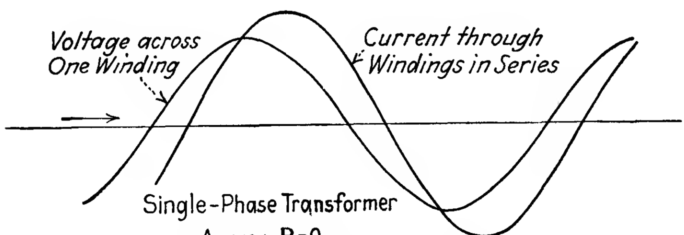

FIG. 29.—Oscillogram from leakage-impedance test by the series-opposition method. Note the sinusoidal shape of the voltage and current waves. Circuit connections shown in Fig. 28.

When the applied voltage is sinusoidal, the current flowing, as well as the terminal voltages of the two windings, will also be sinusoidal, since the flux in the core is suppressed. Exact measurements of the two voltages and the current involved are readily obtained, and oscillograms can be omitted. From the standpoint of precision, therefore, this test is very satisfactory. It gives, however, the leakage impedances at *zero saturation*, and the values obtained may be slightly larger than those corresponding to normal density.

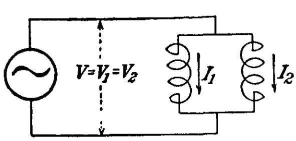

FIG. 30.—Single-phase determination of the ratio of separate leakage impedances by the parallel-conjunction test.

Figure 29 shows an oscillogram taken during a series opposition test of a small laboratory transformer. It will be noted that both voltage and current waves are pure sinusoids.

c. Parallel-conjunction Test.—With the windings excited in parallel conjunction as indicated in Fig. 30 the current will divide

between the two windings in the inverse ratio of their leakage impedances.[1] The equations for this case are

$$V_1 = (R_1 + jX_{11})I_1 + jX_{12}I_2 + E_c \qquad (104)$$
$$V_2 = (R_2 + jX_{22})I_2 + jX_{21}I_1 + E_c \qquad (105)$$

which, by subtraction, gives

$$0 = (R_1 + jX_1)I_1 - (R_2 + jX_2)I_2 = Z_1I_1 - Z_2I_2 \qquad (106)$$

Hence,

$$\frac{I_1}{I_2} = \frac{Z_2}{Z_1} \qquad (107)$$

This test may be performed at any desired value of saturation by varying the impressed voltage. Equation (107) is, in general, applicable to a single frequency only, and, since the currents in this test will always be more or less distorted, oscillographic records are necessary. The oscillograms Figs. 31 and 32, which were taken when tests of this type were applied to a small laboratory transformer, serve to illustrate this point. As seen, prominent harmonics are present even when the saturation is comparatively low (see Fig. 31).

[1] It might be thought that, with the windings in *parallel opposition*, the current would divide inversely as the leakage impedances and also that, with this connection, the impressed voltage would equal each of the leakage impedance drops. This is erroneous, however, as the following analysis will show. The equations for this case are

$$V_1 = (R_1 + jX_{11})I_1 - jX_{12}I_2 + j\omega M_c(I_1 - I_2) \qquad (a)$$
$$V_2 = (R_2 + jX_{22})I_2 - jX_{21}I_1 + j\omega M_c(I_1 - I_2) \qquad (b)$$

Addition of these equations gives

$$V_1 + V_2 = 2V = Z_1I_1 + Z_2I_2 \qquad (c)$$

or

$$V = \frac{Z_1I_1}{2} + \frac{Z_2I_2}{2} \qquad (d)$$

Hence, the impressed voltage is equal to one-half the sum of the separate leakage-impedance drops of the two windings.

By equating equations (*a*) and (*b*), the current division is given by

$$\frac{I_1}{I_2} = \frac{R_2 + j(X_{22} + X_{12}) + 2j\omega M_c}{R_1 + j(X_{11} + X_{21}) + 2j\omega M_c} \qquad (e)$$

which is entirely different from the inverse leakage-impedance ratio.

Only when the two windings are identical in every respect and perfectly symmetrically arranged with respect to the core will the currents divide inversely as the leakage impedances. The ratio of the currents in such a case would evidently be unity, and the windings would have the same impedance. The last term in equations (*a*) and (*b*) would be zero (*i.e.*, the core as a whole completely demagnetized), and the impressed voltage equal to the impedance drop of either winding.

Only when the leakage impedances contain no resistance (or, in practice, when the resistance is entirely negligible as compared to the reactance) can oscillograms be omitted. In this particular case, the ratio of the effective values of the distorted current will

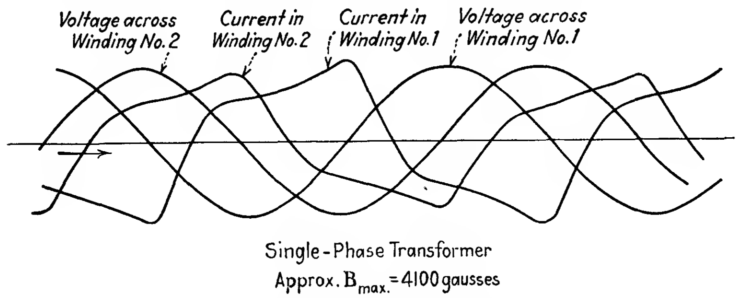

FIG. 31.—Oscillogram from leakage-impedance test by the parallel-conjunction method. Circuit connections shown in Fig. 30. A two-element oscillograph was used, necessitating a separate exposure for each winding. The apparent phase displacements, therefore, between the recorded quantities for winding 1 and winding 2 have no real significance. The voltages should be very nearly in phase. The voltage across winding 1 is reversed with respect to the current and represents a voltage rise instead of a drop.

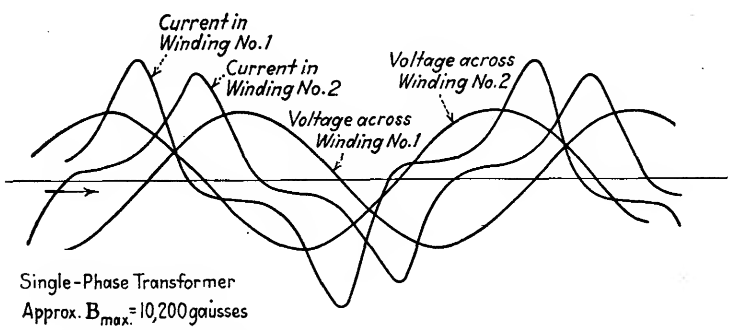

FIG. 32.—Oscillogram from leakage-impedance test by the parallel-conjunction method. Circuit connections shown in Fig. 30. A two-element oscillograph was used, necessitating a separate exposure for each winding. The apparent phase displacements, therefore, between the recorded quantities for winding 1 and winding 2 have no real significance. The voltages should be very nearly in phase. The voltage across winding 1 is reversed with respect to the current and represents a voltage rise instead of a drop. Note that the distortion is larger than shown in Fig. 31 on account of increase in saturation.

equal the inverse ratio of the leakage reactances. Assuming fundamental and third-harmonic components only, the ratios are as follows:

$$\frac{I_1'}{I_2'} = \frac{X_2'}{X_1'} = \frac{3X_2'}{3X_1'} = \frac{X_2'''}{X_1'''} = \frac{I_1'''}{I_2'''} \qquad (108)$$

Hence, the ratio of the fundamental and the third-harmonic currents in the two windings is the same. This would be so also for the fifth harmonics, seventh harmonics, etc. The ratio of the effective currents may be written

$$\frac{I_1}{I_2} = \sqrt{\frac{(I_1')^2 + (I_1''')^2}{(I_2')^2 + (I_2''')^2}} \tag{109}$$

Substituting for I_2' and I_2''' by means of equation (108), equation (109) reduces to

$$\frac{I_1}{I_2} = \sqrt{\frac{(I_1')^2 + (I_1''')^2}{(I_1')^2\left(\frac{X_1'}{X_2'}\right)^2 + (I_1''')^2\left(\frac{X_1'}{X_2'}\right)^2}} = \frac{X_2'}{X_1'} \tag{110}$$

which indicates that the effective currents divide inversely as the leakage reactances.

The parallel-conjunction test gives the ratio only of the leakage impedances and not their actual values. When the equivalent impedance is known from other tests, however, it can be apportioned between the two windings, and the separate leakage impedances determined in this manner.

Three-phase Tests.—The three-phase method by which the separate leakage reactance of the windings of a transformer may be determined makes use of the third-harmonic component which inherently exists in the magnetizing current of a transformer when a sinusoidal voltage is impressed. The method is applicable only when a three-phase bank of three identical transformers is available.

The principle of the method is as follows: If sinusoidal voltages are impressed on a Y-Δ-connected bank of transformers, the third-harmonic component of the magnetizing current will be confined to the delta, where it appears as a circulating current. If the transformers are perfectly balanced and there is no external load on the secondary, no current other than the third harmonic and its multiples can exist in the delta. Usually, the ninth and fifteenth harmonics, etc., are negligible and need not be considered. The third-harmonic electromotive force induced per phase of the delta is just balanced by the triple-frequency impedance drop due to the circulatory third-harmonic current. The problem is then to measure with precision the proper third-harmonic electromotive force and current, which, by simple division, will give the desired triple-frequency leakage impedance.

The three-phase third-harmonic tests determine the leakage reactances at normal (or any desired) saturation. They should give good accuracy of measurement. Of course, due to unbalance and other causes, it may be impracticable to obtain entirely pure waves; but, in any event, the quantities of triple frequency which it is desired to measure will be entirely predominant, and, hence, correct determination is highly facilitated even though oscillogram analysis may be necessary. Furthermore, instrument transformers, if used, are not likely to affect the results seriously, since their secondaries are connected directly to indicating meters. All doubt in regard to the calibration of the instrument transformers is thus eliminated.

a. Two-winding Transformers.—The bank is connected Y-Δ, and balanced sinusoidal voltages impressed. The third-harmonic current in the delta and the third-harmonic electromotive force per phase on the primary side are recorded. The latter is most

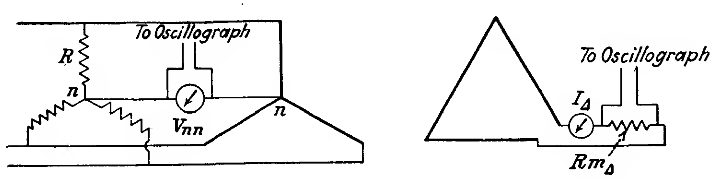

FIG. 33.—Diagram of connections for three-phase leakage-impedance test on two-winding transformers.

practically obtained by connecting a Y-connected resistor bank between the lines and measuring the voltage between the resistor and transformer neutrals. The resistance of the voltmeter and the bank of resistors should be sufficiently high so that the primary third-harmonic current is negligible compared to the current circulating in the delta.

If the generator is Y-connected and its phase voltage is free from a third harmonic (and multiples), the bank of resistors may be omitted and the voltage measured between generator and transformer neutrals. No commercial Y-connected generator, however, is entirely without a third-harmonic component in its voltage to neutral, so this method is scarcely of practical interest.

As a rule, it will be necessary to take oscillographic records and separate out the third harmonics by analysis. While the current

in the delta is sensibly third harmonic, a fundamental and also other harmonics are unavoidable between the two neutrals, if even the slightest unbalance in the impressed voltages, the resistors, or the transformers themselves is present.

The diagram of connections and the third-harmonic vector diagram are given in Figs. 33 and 34, respectively. One to one ratio

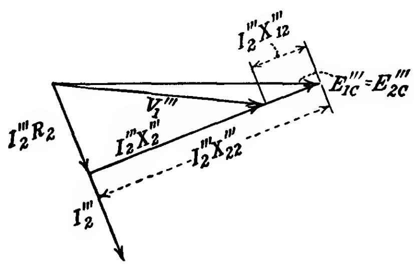

FIG. 34.—Vector diagram of third-harmonic quantities involved in the "two-winding method."

of transformation is assumed. If the transformers have another ratio, the quantities in the various equations given below should all be referred to the same side. Figure 35 shows an oscillogram taken during a test of a bank of experimental transformers.

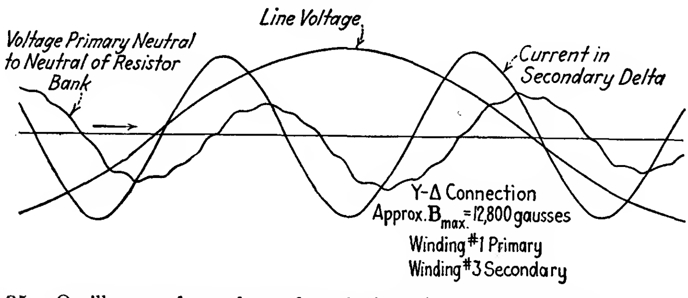

FIG. 35.—Oscillogram from three-phase leakage-impedance test on two-winding transformers. Circuit connections shown in Fig. 33.

If R is the resistance of the resistors per phase, r the resistance of the voltmeter, and V'''_{nn} the third-harmonic voltage between the neutrals, then

$$V_1''' = V'''_{nn}\left(1 + \frac{R}{3r}\right) \tag{111}$$

Since the third-harmonic current in the primary is zero (or, at least, negligibly small), this voltage is the sum of two compo-

nents only. One component E_c''' is induced in each primary winding by the triple-frequency flux ϕ_c''' in the core. The second component $jX_{12}'''I_2'''$ is induced by the part of the third-harmonic flux in the air which produces linkages with the primary winding. Applying equations (28) and (29) to third-harmonic quantities, remembering that I_1''' and V_2''' are both zero, gives

$$\begin{aligned} V_1''' &= jX_{12}'''I_2''' - (R_2 + jX_{22}''')I_2''' \\ &= -(R_2 + jX_2''')I_2''' = -Z_2'''I_2''' \end{aligned} \tag{112}$$

The triple-frequency leakage impedance and reactance of the secondary windings are now found by

$$Z_2''' = -\frac{V_1'''}{I_2'''} \tag{113}$$

$$X_2''' = \sqrt{(Z_2''')^2 - (R_2)^2} \tag{114}$$

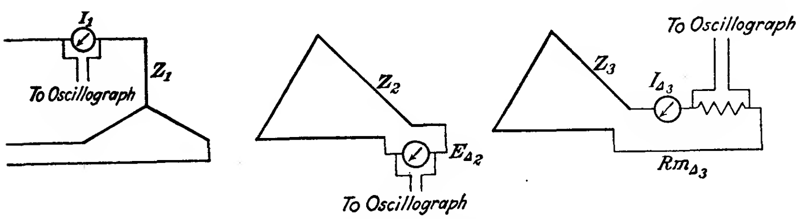

FIG. 36.—Diagram of connections for three-phase leakage-impedance test on three-winding transformers.

This X_2''' is the *true triple-frequency leakage reactance.* By repeating the measurements with the original primary winding as secondary, and *vice versa,* the leakage reactance of the other winding may be found in a similar manner. Dividing the triple-

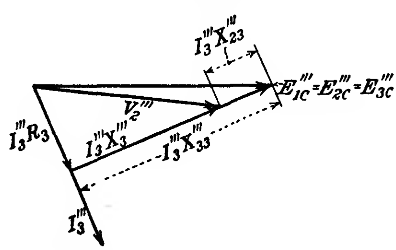

FIG. 37.—Vector diagram of third-harmonic quantities involved in the "three-winding method."

frequency reactances by three, the fundamental reactances are found.

b. Multi-winding Transformers.—When the transformers have more than two windings, a more convenient method may be used

which eliminates the necessity of establishing the artificial neutral on the primary side. As before, the primaries are Y-connected, while the other two windings are Δ-connected. One delta is closed, and the circulating third-harmonic current in it recorded. The other delta is not closed; the third-harmonic voltage appearing across the open corner of this delta may, therefore, be measured and will, for balanced conditions, be equal to three times the third-harmonic electromotive force per phase. Recording these two quantities makes it possible to determine the leakage react-

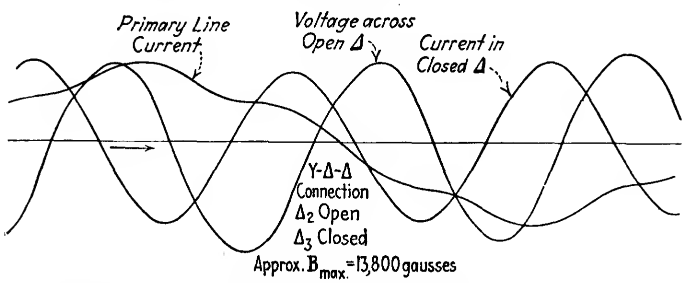

FIG. 38.—Oscillogram from three-phase leakage-impedance test on three-winding transformers. Circuit connections shown in Fig. 36.

ance of the closed delta winding with respect to the open delta winding. In many cases, oscillographic records are unnecessary when the transformers are well balanced, as both the voltage and the current will be sensibly third harmonic.

The connections are shown in Fig. 36, and the third-harmonic vector diagram in Fig. 37. An oscillogram from a test at this type is shown in Fig. 38.

Solution of the vector diagram exactly as in the preceding case gives the leakage reactance of winding 3 with respect to winding 2. By repeating the measurements with changed connections, the other individual reactances are obtained.

EXAMPLE 2

Statement of Problem

Three small three-circuit transformers having unity ratio of transformation are used in a research laboratory. The windings are designated 1, 2, and 3. It was desired to determine the separate leakage reactance of winding 3 with respect to winding 2.

In order to do this, the transformers were connected as in Fig. 39. Power was supplied to the primary Y-connected winding and oscillograms taken of the primary line current, of the voltage appearing across the open corner of

winding 2, and of the current circulating in winding 3. Harmonic analysis of the oscillograms by the 11-coordinate schedule method[1] gave the following results:

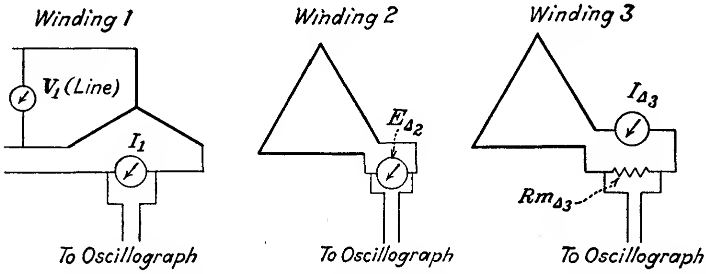

FIG. 39.—Diagram of connections for three-phase leakage-impedance tests considered in Example 2.

SINE COEFFICIENTS
Expressed in Arbitrary Units

Curve analyzed	A_1	A_3	A_5	A_7	A_9	A_{11}
I_1	31.09	0.55	3.42	−0.38	0.08	0.02
$I_{\Delta 3}$	2.77	32.53	0.89	−0.05	−0.20	0.10
$E_{\Delta 2}$	−9.29	37.87	0.56	−0.21	1.20	−0.19

COSINE COEFFICIENTS
Expressed in Arbitrary Units

Curve analyzed	B_1	B_3	B_5	B_7	B_9	B_{11}
I_1	0.23	0.16	−0.89	0.30	0.11	0.10
$I_{\Delta 3}$	0.66	−1.41	−0.01	0.28	0.41	0.08
$E_{\Delta 2}$	0.15	−0.99	−0.61	0	1.63	−0.17

RESULTANT COEFFICIENTS
Expressed in Arbitrary Units

Curve analyzed	C_0	C_1	C_3	C_5	C_7
I_1	31.30	31.10	0.57	3.53	0.48
$I_{\Delta 3}$	32.70	2.85	32.60	0.89	0.28
$E_{\Delta 2}$	39.20	9.29	37.88	0.63	0.21

HARMONICS IN PER CENT OF EQUIVALENT SINE WAVE

Curve analyzed	C_0	C_1	C_3	C_5	C_7
I_1	100	99.4	1.83	11.28	1.54
$I_{\Delta 3}$	100	8.71	99.7	2.72	0.87
$E_{\Delta 2}$	100	29.8	96.4	1.60	0.54

[1] LIPKA, JOSEPH, "Graphical and Mechanical Computation," Chap. VII, John Wiley & Sons, Inc., New York, 1918.

The actually measured values were

$$V_{1\,(\text{line})} = 234 \text{ volts}$$
$$E_{\Delta 2} = 8.32 \text{ volts}$$
$$I_{\Delta 2} = 1.785 \text{ amp.}$$

The resistance of the voltmeter in the corner of winding 2 was sufficiently high so that negligible current would flow in this winding. The total metering resistance in the corner of winding 3, including the resistance of the thermocouple ammeter, was 0.66 ohm. The resistance of winding 3 was 0.466 ohm per phase.

From the above data, calculate the separate leakage reactance of winding 3 with respect to winding 2. Base the calculations on the third-harmonic components and also on meter readings alone and determine the discrepancy between the two values so obtained.

Solution

Third-harmonic voltage across corner of open delta

$$E'''_{\Delta 2} = 8.32 \times 0.964 = 8.02 \text{ volts}$$

Third-harmonic current in closed delta

$$I'''_{\Delta 3} = 1.785 \times 0.997 = 1.781 \text{ amp.}$$

$$\frac{E'''_{\Delta 2}}{3I'''_{\Delta 3}} = \frac{8.02}{3 \times 1.781} = 1.501 \text{ ohms}$$

The resistance part of this impedance obviously includes the resistance per phase of winding 3 as well as one-third of the metering resistance in the corner of this winding. Hence, the triple-frequency reactance becomes

$$x'''_{3(2)} = \sqrt{1.501^2 - \left(0.466 + \frac{0.66}{3}\right)^2} = 1.336 \text{ ohms}$$

The fundamental leakage reactance of winding 3 with respect to winding 2 is, therefore,

$$x'_{3(2)} = \frac{1.336}{3} = 0.445 \text{ ohm}$$

Using meter readings alone, the following values are obtained

$$\frac{E_{\Delta 2}}{3I_{\Delta 3}} = \frac{8.32}{3 \times 1.785} = 1.554 \text{ ohms}$$

$$x'''_{3(2)} = \sqrt{1.554^2 - \left(0.466 + \frac{0.66}{3}\right)^2} = 1.395 \text{ ohms}$$

$$x'_{3(2)} = \frac{1.395}{3} = 0.465 \text{ ohm}$$

Discrepancy between the two values

$$\frac{465 - 0.445}{0.445} 100 = 4.50 \text{ per cent}$$

The discrepancy, as seen, is not large. This shows that, when the impressed voltages and the transformers themselves are well balanced, fair accuracy may be obtained from meter readings alone without resorting to oscillogram analysis.

EXAMPLE 3

Statement of Problem

In order to determine the possible variation with saturation of the leakage reactance of winding 3 with respect to winding 2 of the laboratory transformers referred to in the preceding example and to obtain, if necessary, an average value, additional tests were carried out (see Fig. 39). These were performed at four values of impressed voltage so as to give flux densities of from 11,000 to 14,000 gausses approximately. As seen, this range covers what might be termed the *operating region* of the transformers.

The data obtained are given in Table V.

TABLE V

Connections	Oscillogram number	Approximate flux density, gausses	Line volts impressed	Volts across open delta $E_{\Delta 2}$	Amperes in closed delta $I_{\Delta 3}$	Third-harmonic volts across open delta $E'''_{\Delta 2}$	Third-harmonic amperes in closed delta $I'''_{\Delta 3}$	Ohms resistance in metering circuit $R_{m\Delta 3}$
Winding 1, primary Y.	1	11,300	192.0	3.22	0.691	3.10	0.689	0.66
Winding 2,	2	12,300	210.0	4.60	0.987	4.43	0.984	0.66
open Δ. Winding 3,	3	13,100	222.0	6.10	1.310	5.88	1.308	0.66
closed Δ.	4	13,800	234.0	8.32	1.785	8.02	1.781	0.66

Calculate the average leakage reactance of winding 3 with respect to 2, using third-harmonic quantities as well as meter readings alone. Compare average and individual values.

Solution

Table VI gives the results calculated by the process indicated in Example 2.

TABLE VI

Oscillogram number	Line volts impressed	Approximate flux density, gausses	Ohms impedance $\dfrac{E_{\Delta 2}}{3I_{\Delta 3}}$	Ohms impedance $\dfrac{E'''_{\Delta 2}}{3I'''_{\Delta 3}}$
1	192.0	11,300	1.554	1.500
2	210.0	12,300	1.553	1.502
3	222.0	13,100	1.552	1.500
4	234.0	13,800	1.554	1.501
Average.....	......	......	1.553	1.501

It appears from the table that the four values of leakage impedance (including metering resistance) practically coincide. This holds for the figures obtained by using third-harmonic quantities as well as for those obtained from meter readings alone. Hence, the reactance values obtained by eliminating the resistance part from the impedances would also coincide. These results, therefore, substantiate the statement previously made in this chapter that the change in leakage reactance is negligibly small for a reasonable variation of saturation in the operating range.

Inspection of the figures shows that the average leakage reactance would be the same as calculated from a single test in Example 2.

CHAPTER III

UNBALANCED CIRCUITS

In the operation of polyphase systems, it is usually attempted to keep all loads and, hence, the voltages and currents as nearly balanced as possible. In spite of this fact, there are systems where quite severe unbalance may exist. This may particularly be the case where heavy single-phase loads, such as electric furnaces, are connected. Unbalance is also caused by disymmetrical short circuits.

The solution of problems involving unbalanced impedances, voltages, and currents is frequently quite laborious. When sufficient data are at hand, such problems can be solved by the application of Ohm's and Kirchhoff's laws. This method, however, is often inconvenient and may even lead to considerable difficulties when rotating machines are involved.

A much more convenient scheme is the method of symmetrical coordinates. This method, which is due to C. L. Fortescue,[1] is an extremely powerful one for handling problems of unbalance. It depends upon the fact that in any n-phase system the actual sinusoidal unbalanced voltages and currents may, in general, be replaced by $n - 1$ symmetrical n-phase systems of voltages and currents and one single-phase system. The angular displacement between the vectors belonging to consecutive phases will be $k(2\pi/n)$ in the kth of the $n - 1$ symmetrical systems. Hence, if the circuit has an odd number of phases (n is an odd integer), the $n - 1$ symmetrical systems will all be *balanced n-phase systems.* The phase sequence of these balanced systems will, of course, not be the same. The essential point, however, is that they really are balanced systems and may be treated as such. On the other hand, if the circuit has an even number of phases (n is an even integer), the $n - 1$ symmetrical systems will *not all be balanced n-phase systems.* In many cases, the circuit connections are such that the single-phase system disappears, leaving only the symmetrical polyphase systems to be considered.

[1] FORTESCUE, C. L., "Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks," *Trans. A.I.E.E.*, p. 1027, 1918.

The general problem of unbalance in an n-phase system will not be treated here, since such systems are not encountered in practice. The discussion of the method of symmetrical coordinates will be confined to three-phase, two-phase, and four-phase circuits. This will cover most practical cases and will also illustrate the fundamental principles sufficiently well, so that no particular difficulty should be encountered in extending the theory to other polyphase systems.

Solution by Simultaneous Equations Based on Ohm's and Kirchhoff's Laws.—This method will be illustrated by application to a few typical three-phase cases. Exactly the same principles, however, may be used in connection with other polyphase systems, and solutions for such systems are readily worked out on the same basis.

All impedances involved are assumed constant and independent of the current unbalance. If some of the impedances vary with

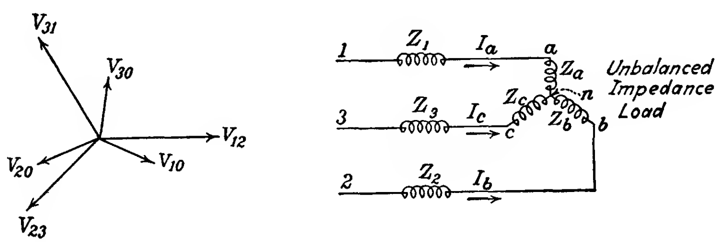

FIG. 40.—Unbalanced three-phase Y-connected impedance load on unbalanced feeder. No neutral connection.

the amount of current unbalance, they must be given the value actually corresponding to the particular conditions existing in the circuit. Often these impedances cannot be readily ascertained, and, in such cases, the method of solution by simultaneous equations for the several phases fails. For this reason, for instance, it is exceedingly difficult, if not impossible, to handle problems concerning unbalanced operation of rotating machines by this method. Such problems should be analyzed by the method of symmetrical coordinates.

Unbalanced Three-phase Y-connected Load on Unbalanced Feeder.—Consider the circuit shown in Fig. 40. The line voltages at the sending end of the feeder are known, and it is desired to solve for the currents.

By taking voltage drops around the circuits, the following equations may be written:

$$V_{12} = I_a(Z_1 + Z_a) - I_b(Z_2 + Z_b) \tag{1}$$
$$V_{23} = I_b(Z_2 + Z_b) - I_c(Z_3 + Z_c) \tag{2}$$
$$V_{31} = I_c(Z_3 + Z_c) - I_a(Z_1 + Z_a) \tag{3}$$

Since there is no neutral connection, the currents must add to zero. Hence,

$$I_a + I_b + I_c = 0 \tag{4}$$

Combining equations (4) and (1) gives

$$V_{12} = I_a(Z_1 + Z_2 + Z_a + Z_b) + I_c(Z_2 + Z_b) \tag{5}$$

which, upon elimination of I_c by making use of equation (3), becomes

$$V_{12} = I_a\left[Z_1 + Z_2 + Z_a + Z_b + \frac{(Z_1 + Z_a)(Z_2 + Z_b)}{Z_3 + Z_c}\right] + V_{31}\frac{Z_2 + Z_b}{Z_3 + Z_c} \tag{6}$$

or

$$I_a = \frac{V_{12}(Z_3 + Z_c) - V_{31}(Z_2 + Z_b)}{(Z_1 + Z_2 + Z_a + Z_b)(Z_3 + Z_c) + (Z_1 + Z_a)(Z_2 + Z_b)} \tag{7}$$

In a similar manner, the solutions for the currents in phases b and c become

$$I_b = \frac{V_{23}(Z_1 + Z_a) - V_{12}(Z_3 + Z_c)}{(Z_2 + Z_3 + Z_b + Z_c)(Z_1 + Z_a) + (Z_2 + Z_b)(Z_3 + Z_c)} \tag{8}$$

$$I_c = \frac{V_{31}(Z_2 + Z_b) - V_{23}(Z_1 + Z_a)}{(Z_3 + Z_1 + Z_c + Z_a)(Z_2 + Z_b) + (Z_3 + Z_c)(Z_1 + Z_a)} \tag{9}$$

Equations (7), (8), and (9), which, as seen, are symmetrical, give the currents in terms of the line voltage at the sending end of the feeder and the line and load constants. Knowing the currents, the voltages to neutral and the line voltages at the load are readily computed. Thus,

$$V_{an} = I_aZ_a \tag{10}$$
$$V_{bn} = I_bZ_b \tag{11}$$
$$V_{cn} = I_cZ_c \tag{12}$$

and

$$V_{ab} = V_{an} - V_{bn} = I_aZ_a - I_bZ_b \tag{13}$$
$$V_{bc} = V_{bn} - V_{cn} = I_bZ_b - I_cZ_c \tag{14}$$
$$V_{ca} = V_{cn} - V_{an} = I_cZ_c - I_aZ_a \tag{15}$$

The impressed line voltages V_{12}, V_{23}, and V_{31} may be balanced or unbalanced. The formulas so far established are general and applicable to any case. When the line voltages are balanced, the

voltages to neutral may be either balanced or unbalanced. It should be noted in this connection that a balanced system of voltages to neutral will always give rise to balanced line voltages, while evidently the reciprocal is not true. If the impressed voltages to neutral are known, the solution of the problem just discussed may be obtained in a slightly different way. This involves the determination of the displacement of the neutral point of the load from the neutral point corresponding to the applied voltages. Assume that the latter are actually balanced and that the magnitude of the line voltages is V. Then

$$V_{10} = \frac{V}{\sqrt{3}}\overline{\backslash 30^\circ} = I_a(Z_1 + Z_a) + V_{n0} \tag{16}$$

$$V_{20} = \frac{V}{\sqrt{3}}\overline{\backslash 150^\circ} = I_b(Z_2 + Z_b) + V_{n0} \tag{17}$$

$$V_{30} = \frac{V}{\sqrt{3}}\underline{/90^\circ} = I_c(Z_3 + Z_c) + V_{n0} \tag{18}$$

Combining equations (4), (16), (17), and (18), the following solution is obtained for the voltage V_{n0} between the two neutral points:

$$V_{n0} = \frac{V}{\sqrt{3}} \frac{1\overline{\backslash 30^\circ} + \dfrac{Z_1 + Z_a}{Z_2 + Z_b}\overline{\backslash 150^\circ} + \dfrac{Z_1 + Z_a}{Z_3 + Z_c}\underline{/90^\circ}}{1 + \dfrac{Z_1 + Z_a}{Z_2 + Z_b} + \dfrac{Z_1 + Z_a}{Z_3 + Z_c}} \tag{19}$$

When the voltage V_{n0} is determined, the currents are obtained from equations (16), (17), and (18) as follows:

$$I_a = \frac{\dfrac{V}{\sqrt{3}}\overline{\backslash 30^\circ} - V_{n0}}{Z_1 + Z_a} \tag{20}$$

$$I_b = \frac{\dfrac{V}{\sqrt{3}}\overline{\backslash 150^\circ} - V_{n0}}{Z_2 + Z_b} \tag{21}$$

$$I_c = \frac{\dfrac{V}{\sqrt{3}}\underline{/90^\circ} - V_{n0}}{Z_3 + Z_c} \tag{22}$$

Having obtained the solutions for the currents, the load voltages are calculated as previously indicated. In other words, equations (10) to (15) inclusive are used.

When the feeder impedances and the load impedances are symmetrical, there is obviously no displacement of the load neutral with respect to the neutral point corresponding to the

impressed voltages. This is in accordance with equation (19), where the impedance combination in the numerator equals zero for this condition.

Assume next that there is a neutral connection; *i.e.*, consider the circuit shown in Fig. 41. In order to solve for the currents in this circuit, knowledge of the impressed line voltages alone does not suffice, since an additional unknown, namely, the current in the neutral, is introduced. The currents in the three phases no longer add to zero but are equal to the neutral current reversed. Thus,

$$I_a + I_b + I_c + I_n = 0 \tag{23}$$

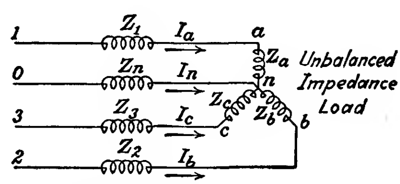

FIG. 41.—Unbalanced three-phase Y-connected impedance load on unbalanced feeder. Neutral connection.

It is necessary, in this case, to know the impressed voltages to neutral or else to have some quantity given besides the line voltages. In working out the solution below, however, it will be assumed that the *impressed voltages to neutral are strictly balanced.* This is the same assumption that was made in the second solution of the circuit without a neutral connection.

The voltage between the two neutrals V_{n0} is now given by

$$V_{n0} = -I_n Z_n \tag{24}$$

Evidently, equations (16), (17), and (18) still hold, and by combining these with equations (23) and (24) the following expression is obtained for the neutral current:

$$I_n = -\frac{V}{\sqrt{3}} \frac{1\diagdown 30^\circ + \dfrac{Z_1 + Z_a}{Z_2 + Z_b}\diagdown 150^\circ + \dfrac{Z_1 + Z_a}{Z_3 + Z_c}\diagup 90^\circ}{Z_1 + Z_a + Z_n\left[1 + \dfrac{Z_1 + Z_a}{Z_2 + Z_b} + \dfrac{Z_1 + Z_a}{Z_3 + Z_c}\right]} \tag{25}$$

Upon determining the current in the neutral, the voltage V_{n0} is calculated by equation (24), and the phase currents obtained by equations (20) to (22) inclusive. The load voltages are then determined as when the neutrals are isolated.

Unbalanced Three-phase Δ-connected Load on Unbalanced Feeder.—When the unbalanced load is Δ-connected, as in Fig. 42, the following equations may immediately be written down:

$$V_{12} = I_a Z_1 + I_{ab} Z_{ab} - I_b Z_2 \tag{26}$$
$$V_{23} = I_b Z_2 + I_{bc} Z_{bc} - I_c Z_3 \tag{27}$$
$$V_{31} = I_c Z_3 + I_{ca} Z_{ca} - I_a Z_1 \tag{28}$$
$$I_{ab} Z_{ab} + I_{bc} Z_{bc} + I_{ca} Z_{ca} = 0 \tag{29}$$
$$I_a = I_{ab} - I_{ca} \tag{30}$$
$$I_b = I_{bc} - I_{ab} \tag{31}$$
$$I_c = I_{ca} - I_{bc} \tag{32}$$

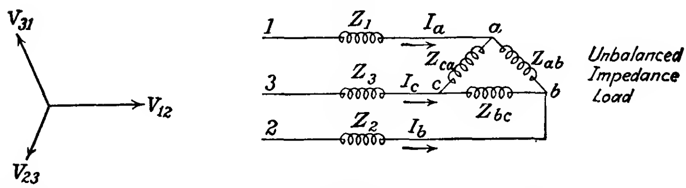

FIG. 42.—Unbalanced three-phase Δ-connected impedance load on unbalanced feeder.

By simultaneous solution of equations (29) to (32) inclusive, expressions for the load currents in terms of the line currents are obtained.

$$I_{ab} = \frac{I_a Z_{ca} - I_b Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}} \tag{33}$$
$$I_{bc} = \frac{I_b Z_{ab} - I_c Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}} \tag{34}$$
$$I_{ca} = \frac{I_c Z_{bc} - I_a Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}} \tag{35}$$

Substituting for the load currents in equations (26), (27), and (28), these reduce to

$$V_{12} = I_a\left(Z_1 + \frac{Z_{ca}Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}}\right) - I_b\left(Z_2 + \frac{Z_{ab}Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}}\right) \tag{36}$$

$$V_{23} = I_b\left(Z_2 + \frac{Z_{ab}Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}}\right) - I_c\left(Z_3 + \frac{Z_{bc}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}\right) \tag{37}$$

$$V_{31} = I_c\left(Z_3 + \frac{Z_{bc}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}\right) - I_a\left(Z_1 + \frac{Z_{ca}Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}}\right) \tag{38}$$

Comparison of these equations with equations (1), (2), and (3) for the unbalanced Y-connected load without neutral shows complete similarity in form. As a matter of fact, equations (36), (37), and (38) might have been written down immediately *if the unbalanced Δ were replaced by its equivalent Y-circuit.* Obviously, the impedance combinations in (36), (37), and (38) are just the equivalent Y-impedances of the delta, *viz.*,

$$Z_a = \frac{Z_{ca}Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}} \tag{39}$$

$$Z_b = \frac{Z_{bc}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}} \tag{40}$$

$$Z_c = \frac{Z_{ca}Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}} \tag{41}$$

It seems, therefore, that the most convenient method of attack is to replace the Δ by its equivalent Y and solve the problem on this basis. Hence, the line currents are given by equations (7) to (9) inclusive.

After having determined the line currents, the load currents (Δ-currents) are calculated by equations (33), (34), and (35). Finally, the load voltages are obtained by

$$V_{ab} = I_{ab}Z_{ab} \tag{42}$$

$$V_{bc} = I_{bc}Z_{bc} \tag{43}$$

$$V_{ca} = I_{ca}Z_{ca} \tag{44}$$

EXAMPLE 1

Statement of Problem

Two identical 100-kv.-a., 11,000/460-volt, 60-cycle transformers are operated in V-connection (Fig. 43). The following short-circuit data apply to each transformer:

Volts	Amperes	Watts
265	9.1	1,004

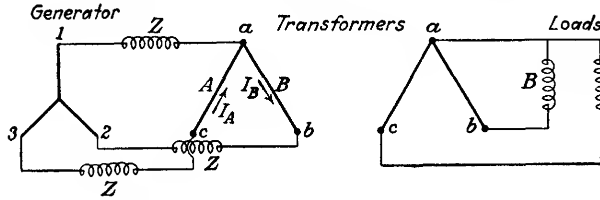

FIG. 43.—Diagram of V-connected transformers for Example 1.

The transformers are fed from a Y-connected generator over feeders having a resistance and a reactance per conductor of 1.0 ohm and 2.0 ohms, respectively. The generator is of sufficient capacity so that its terminal voltages may be assumed to remain constant and balanced at 11,200 volts between lines.

The transformers supply power to two single-phase loads drawing currents of 119.5 amp. at 90 per cent power factor (lagging) and 71.7 amp. at 80 per cent power factor (lagging), respectively. Determine the voltages at the loads and the power delivered to each. Assume that V_{12} leads V_{31}.

Solution

Ratio of Transformation:

$$a = \frac{11{,}000}{460} = 23.9$$

Transformer constants referred to primary

$$Z_e = \frac{265}{9.1} = 29.1 \text{ ohms}$$

$$R_e = \frac{1{,}004}{9.1^2} = 12.1 \text{ ohms}$$

$$X_e = \sqrt{29.1^2 - 12.1^2} = 26.5 \text{ ohms}$$

Let V_A and V_B be the secondary voltages and I_A and I_B the secondary currents, all referred to the primary side. Then

$$V_{ab} = V_B + I_B Z_e \qquad (a)$$

$$V_{ca} = V_A + I_A Z_e \qquad (b)$$

Taking the current directions as shown in the diagram, the generator voltages may be written

$$\begin{aligned} V_{31} &= V_{ca} + (I_{a1} + I_{3c})Z \\ &= V_{ca} + (2I_A - I_B)Z \\ &= V_A + I_A(Z_e + 2Z) - I_B Z \qquad (c) \\ V_{12} &= V_B + I_B(Z_e + 2Z) - I_A Z \qquad (d) \\ Z &= 1.0 + j2.0 = 2.236\underline{/63^\circ.4} \text{ ohms} \end{aligned}$$

$$I_A = \frac{119.5}{23.9} = 5.0 \text{ amp.}$$

$$I_B = \frac{71.7}{23.9} = 3.0 \text{ amp.}$$

V_A will be taken as reference vector. The exact phase displacement between V_A and V_B is not known. It should, however, be in the neighborhood of 120 deg. At present, it will be introduced as θ deg. Thus,

$$V_A = V_A\underline{/0} = V_A + j0 \qquad (e)$$

$$V_B = V_B\underline{/\theta} = V_B\,(\cos\theta + j\sin\theta) \qquad (f)$$

$$\begin{aligned} I_A &= 5.0\overline{\backslash 25^\circ.8} = 4.50 - j2.18 \\ I_B &= 3.0\underline{/\theta - 36^\circ.9} \\ &= 3.0[\cos(\theta - 36^\circ.9) + j\sin(\theta - 36^\circ.9)] \\ &= 3.0[0.8\cos\theta + 0.6\sin\theta + j(0.8\sin\theta - 0.6\cos\theta)] \end{aligned}$$

Making these substitutions in equations (c) and (d) gives

$$\begin{aligned} V_{31} &= V_A + j0 + (4.50 - j2.18)(14.1 + j30.5) - \\ &\qquad 3.0\underline{/\theta - 36^\circ.9} \times 2.236\underline{/63^\circ.4} \\ &= (V_A + 2.99\sin\theta - 6.0\cos\theta + 130.7) - \\ &\qquad j(6.0\sin\theta + 2.99\cos\theta - 108.2) \qquad (g) \\ V_{12} &= V_B\,(\cos\theta + j\sin\theta) + 3.0\underline{/\theta - 36^\circ.9}(14.1 + j30.5) - \\ &\qquad (4.5 - j2.18)(1.0 + j2.0) \\ &= (V_B\cos\theta + 88.6\cos\theta - 47.7\sin\theta - 8.91) + \\ &\qquad j(V_B\sin\theta + 47.7\cos\theta + 88.6\sin\theta - 6.82) \qquad (h) \end{aligned}$$

These equations will now be solved by a trial-and-error process. Assume at first that θ is exactly 120 deg. Then

$$\sin\theta = 0.866 \qquad \cos\theta = -0.5$$

Substituting these values in equation (g) gives

$$V_{31} = 11{,}200\underline{/\alpha} = (V_A + 136.3) + j104.5$$
$$11{,}200^2 = (V_A + 136.3)^2 + 104.5^2$$

from which

$$V_A = 11{,}063.2 \text{ volts}$$

With this value of V_A, the vector expression for V_{31} becomes

$$V_{31} = 11{,}199\ 5 + j104.5 = 11{,}200\underline{/0°.6} \text{ volts}$$

Similar substitution in equation (h) gives

$$V_{12} = 11{,}200\underline{/\beta} = (-0.5V_B - 94.5) + j(0.866V_B + 46.0)$$
$$11{,}200^2 = V_B^2 + 174.2V_B + 94.5^2 + 46.0^2$$

from which

$$V_B = 11{,}112.7 \text{ volts}$$

Using this value of V_B, the vector expression for V_{12} becomes

$$V_{12} = -5{,}650.9 + j9{,}669.6 = 11{,}200\underline{/120°.3} \text{ volts}$$

The calculated phase displacement between V_{12} and V_{31} is

$$\beta - \alpha = 120.3 - 0.6 = 119.7 \text{ deg.}$$

This angle, however, should be exactly 120 deg., since the generator voltages are balanced. The original assumption in regard to the phase angle between V_A and V_B, therefore, must be modified. As a second trial, assume $\theta =$ 120.3 deg.

Hence,

$$\sin\theta = 0.8634 \qquad \cos\theta = -0.5045$$
$$V_{31} = 11{,}200\underline{/\alpha} = (V_A + 136.3) + j104.5$$
$$V_A = 11{,}063.2 \text{ volts}$$
$$V_{31} = 11{,}199.5 + j104.5 = 11{,}200\underline{/0°.6} \text{ volts}$$

and

$$V_{12} = 11{,}200\underline{/\beta} = (-0.5045V_B - 94.8) + j(0.8634V_B + 45.6)$$
$$11{,}200^2 = V_B^2 + 174.4V_B + 94.8^2 + 45.6^2$$

from which

$$V_B = 11{,}112.6 \text{ volts}$$

The vector expression for V_{12} now becomes

$$V_{12} = -5701.1 + j9640.2 = 11{,}200\underline{/120°.6} \text{ volts}$$

The calculated phase displacement between V_{12} and V_{31} is now

$$\beta - \alpha = 120.6 - 0.6 = 120 \text{ deg.}$$

which is exactly correct. Hence, it is safe to assume that the values of the voltages V_A and V_B, as found by the second trial, are correct.

Referring these values to the low-tension (secondary) side gives for the load voltages

$$V_A = \frac{11{,}063.2}{23.9} = 462.5 \text{ volts}$$
$$V_B = \frac{11{,}112.6}{23.9} = 465 \text{ volts}$$

The amounts of power absorbed by the two loads are, hence,

$$P_A = \frac{462.5 \times 119.5 \times 0.9}{1{,}000} = 49.8 \text{ kw.}$$
$$P_B = \frac{465 \times 71.7 \times 0.8}{1{,}000} = 26.7 \text{ kw.}$$

Method of Symmetrical Phase Coordinates. General.—Consider the voltages to neutral in a three-phase system. Assume that these voltages are unbalanced and do not add to zero. Although the sum of the line voltages in a three-phase system is always zero whether the system is balanced or not, this is not the case with the phase voltages. Hence, in the general unbalanced case, the sum of the voltages to neutral, $V_a + V_b + V_c$, has a definite value different from zero.

Evidently, each one of these voltages may be resolved into any number of arbitrary components, provided only the sum of the components is equal to the particular voltage which they represent. Nothing at all is gained, however, by such resolution, unless the component voltage systems can be handled more readily than the original unbalanced voltage system itself. It will now be shown that the unbalanced voltages whose sum is different from zero may be replaced by two balanced three-phase systems of opposite phase sequence and one single-phase system. This is the most appropriate resolution and gives a maximum of convenience of handling, since, as a rule, most circuits are susceptible to easy solution where nothing but balanced three-phase or single-phase quantities are involved.

Thus, the unbalanced voltages may each be written in terms of three components as follows:

$$V_a = x + y + z \tag{45}$$

$$V_b = ax + by + cz \tag{46}$$

$$V_c = dx + ey + fz \tag{47}$$

It will be noted that the components of V_b and V_c have been written as a factor times the corresponding components of V_a. This is perfectly permissible and does not reduce the generality of the resolution if the coefficients a, b, etc., are considered to be complex quantities which, at the present, may have any arbitrary magnitude and angle consistent with each equation's being separately satisfied. Mr. Fortescue, in his exposition of the method of symmetrical coordinates applied to n-phase systems, calls these coefficients *sequence operators* and determines their values for resolution into symmetrical polyphase and single-phase systems.

Assume that the sum of the three voltages is $3V_0$. Hence,

$$V_a + V_b + V_c = (1 + a + d)x + (1 + b + e)y + (1 + c + f)z = 3V_0 \tag{48}$$

or

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c) \tag{49}$$

If this quantity V_0 is subtracted from each of the three voltages, it is evident that the remaining vectors will form a closed triangle. Thus,

$$(V_a - V_0) + (V_b - V_0) + (V_c - V_0) = 0 \tag{50}$$

This is equivalent to considering a single-phase (uniphase or zero-sequence) component V_0 to be present in each of the original voltages. Let x be this component. Consequently,

$$x = V_0 = \tfrac{1}{3}(V_a + V_b + V_c) \tag{51}$$

and since the single-phase components in the three phases are considered equal, the zero-sequence operators are all unity, *viz.*,

$$a = 1 \quad \text{and} \quad d = 1 \tag{52}$$

From equations (48) and (50), it now follows that

$$(1 + b + e)y + (1 + c + f)z = 0 \tag{53}$$

In other words, the sum of the y and z components in the three phases must always be zero. This will always hold true when these components each form balanced systems. If these sytems, however, are given the same phase sequence, their resultant (*i.e.*, the system whose phase voltages are $y + z$, $by + cz$, and $ey + fz$) will always be a balanced system. Obviously, the subtraction of the zero-sequence voltage from the original unbalanced voltages does not necessarily result in the voltages so obtained forming a balanced system. In general, it will remain unbalanced. In order to take care of this, the phase sequence of the balanced y and z systems must be taken opposite. The sequence operators, therefore, for these systems become

$$b = 1\overline{\backslash 120^\circ} \quad \text{and} \quad e = 1/\underline{120^\circ} \tag{54}$$

$$c = 1/\underline{120^\circ} \quad \text{and} \quad f = 1\overline{\backslash 120^\circ} \tag{55}$$

The y-system is called the *positive-sequence* (or *direct-phase*) *system* and has the same phase order as the original unbalanced vectors.[1] This assumes that the phase order of the latter is a–b–c in a counterclockwise direction, an assumption which will be adhered to throughout this discussion. The z-system is termed the *negative-sequence* (or *reverse-phase*) *system* and has a phase order opposite to that of the original vectors. When the phase sequence of the two balanced component systems is opposite and the relative size and phase of their vectors is properly adjusted, it is evident that the superposition of the two will give any

[1] The phase order of the original vectors may appear uncertain when the zero-sequence component is considerable. In such cases, the correct phase order is the same as that of the vectors found by subtracting the zero-sequence component from each of the original vectors.

desired system of unbalanced voltages whose sum is zero. The problem is next to determine these components when the unbalanced voltages are known.

The symbol V_0 has already been introduced for the x-system or zero-sequence components. The symbols V^+ and V^- with appropriate subscripts representing phase will be used for the y- and z-systems, or positive- and negative-sequence components, respectively. Equations (45), (46), and (47) may, therefore, be written in a more convenient nomenclature, as follows:

$$V_a = V_a^+ + V_a^- + V_0 \tag{56}$$
$$V_b = V_b^+ + V_b^- + V_0 \tag{57}$$
$$V_c = V_c^+ + V_c^- + V_0 \tag{58}$$

or

$$V_a = V_a^+\underline{/0} + V_a^-\underline{/0} + V_0 \tag{59}$$
$$V_b = V_a^+\overline{\backslash 120^\circ} + V_a^-\underline{/120^\circ} + V_0 \tag{60}$$
$$V_c = V_a^+\underline{/120^\circ} + V_a^-\overline{\backslash 120^\circ} + V_0 \tag{61}$$

A system of unbalanced currents may be treated exactly as a system of unbalanced voltages. Hence, an unbalanced current system may, in general, be split into positive-, negative-, and zero-sequence components, as indicated in the following equations:

$$I_a = I_a^+ + I_a^- + I_0 \tag{62}$$
$$I_b = I_b^+ + I_b^- + I_0 \tag{63}$$
$$I_c = I_c^+ + I_c^- + I_0 \tag{64}$$

or

$$I_a = I_a^+\underline{/0} + I_a^-\underline{/0} + I_0 \tag{65}$$
$$I_b = I_a^+\overline{\backslash 120^\circ} + I_a^-\underline{/120^\circ} + I_0 \tag{66}$$
$$I_c = I_a^+\underline{/120^\circ} + I_a^-\overline{\backslash 120^\circ} + I_0 \tag{67}$$

The zero-sequence component of current is given by

$$I_0 = \tfrac{1}{3}(I_a + I_b + I_c) \tag{68}$$

So far, the resolution of three-phase voltages and currents only has been considered. Applying a similar reasoning, however, to two-phase and four-phase circuits, general equations for the symmetrical phase components of such systems may also be established.

In the two-phase system, the unbalanced voltages and currents may be resolved either into one single-phase and one balanced two-phase system of voltages and currents or into two balanced two-phase systems of opposite phase order. The latter resolution is preferable.

It should be noted that, if the former resolution is used, the single-phase component is not found by taking one-half the sum of the unbalanced vectors. This zero-sequence component, therefore, is not determined by the same general rule which applies to three-phase and other polyphase systems.

Since the resolution into two balanced two-phase systems is generally used, the statement is often made that there can be no zero-sequence components in a two-phase system independent of whether a three-conductor or a four-conductor circuit is used. The unbalanced two-phase voltages V_a and V_b may then be resolved as follows:

$$V_a = V_a^+ + V_a^- = V_a^+\underline{/0} + V_a^-\underline{/0} \qquad (69)$$

$$V_b = V_b^+ + V_b^- = V_b^+\overline{\backslash 90^\circ} + V_a^-\underline{/90^\circ} \qquad (70)$$

The current components are given by

$$I_a = I_a^+ + I_a^- = I_a^+\underline{/0} + I_a^-\underline{/0} \qquad (71)$$

$$I_b = I_b^+ + I_b^- = I_a^+\overline{\backslash 90^\circ} + I_a^-\underline{/90^\circ} \qquad (72)$$

The most general four-phase (or quarter-phase) system has a neutral, in which case, zero-sequence components may be present in the voltages to neutral as well as in the currents. Since in a four-phase system there will be three symmetrical component systems, an additional symbol must be added to the previous notation of plus and minus to indicate the third symmetrical system. An asterisk will be used for this purpose. The general equations for the voltages to neutral may, thus, be written

$$V_a = V_a^+ + V_a^- + V_a^* + V_0 \qquad (73)$$

$$V_b = V_b^+ + V_b^- + V_b^* + V_0 \qquad (74)$$

$$V_c = V_c^+ + V_c^- + V_c^* + V_0 \qquad (75)$$

$$V_d = V_d^+ + V_d^- + V_d^* + V_0 \qquad (76)$$

or

$$V_a = V_a^+\underline{/0} + V_a^-\underline{/0} + V_a^*\underline{/0} + V_0 \qquad (77)$$

$$V_b = V_a^+\overline{\backslash 90^\circ} + V_a^-\underline{/180^\circ} + V_a^*\underline{/90^\circ} + V_0 \qquad (78)$$

$$V_c = V_a^+\underline{/180^\circ} + V_a^-\underline{/0} + V_a^*\underline{/180^\circ} + V_0 \qquad (79)$$

$$V_d = V_a^+\underline{/90^\circ} + V_a^-\underline{/180^\circ} + V_a^*\overline{\backslash 90^\circ} + V_0 \qquad (80)$$

The current equations become

$$I_a = I_a^+ + I_a^- + I_a^* + I_0 \qquad 81)$$

$$I_b = I_b^+ + I_b^- + I_b^* + I_0 \qquad (82)$$

$$I_c = I_c^+ + I_c^- + I_c^* + I_0 \qquad (83)$$

$$I_d = I_d^+ + I_d^- + I_d^* + I_0 \qquad (84)$$

or

$$I_a = I_a^+\underline{/0} + I_a^-\underline{/0} + I_a^*\underline{/0} + I_0 \tag{85}$$

$$I_b = I_a^+\overline{\backslash 90^\circ} + I_a^-\underline{/180^\circ} + I_a^*\underline{/90^\circ} + I_0 \tag{86}$$

$$I_c = I_a^+\underline{/180^\circ} + I_a^-\underline{/0} + I_a^*\underline{/180^\circ} + I_0 \tag{87}$$

$$I_d = I_a^+\underline{/90^\circ} + I_a^-\underline{/180^\circ} + I_a^*\overline{\backslash 90^\circ} + I_0 \tag{88}$$

Analytical Determination of the Components in a Three-phase System.—The zero-sequence voltage is given by equation (49). Substituting this value in equations (59), (60), and (61), these become

$$\tfrac{2}{3}V_a - \tfrac{1}{3}V_b - \tfrac{1}{3}V_c = V_a^+ + V_a^- \tag{89}$$

$$-\tfrac{1}{3}V_a + \tfrac{2}{3}V_b - \tfrac{1}{3}V_c = V_a^+\overline{\backslash 120^\circ} + V_a^-\underline{/120^\circ} \tag{90}$$

$$-\tfrac{1}{3}V_a - \tfrac{1}{3}V_b + \tfrac{2}{3}V_c = V_a^+\underline{/120^\circ} + V_a^-\overline{\backslash 120^\circ} \tag{91}$$

Operating on equation (90) by $\overline{\backslash 120^\circ}$ gives

$$-\tfrac{1}{3}V_a\overline{\backslash 120^\circ} + \tfrac{2}{3}V_b\overline{\backslash 120^\circ} - \tfrac{1}{3}V_c\overline{\backslash 120^\circ} = V_a^+\underline{/120^\circ} + V_a^- \tag{92}$$

By subtracting this from equation (89), V_a^- is eliminated, and the following equation is obtained:

$$\tfrac{2}{3}V_a + \tfrac{1}{3}V_a\overline{\backslash 120^\circ} - \tfrac{1}{3}V_b - \tfrac{2}{3}V_b\overline{\backslash 120^\circ} - \tfrac{1}{3}V_c + \tfrac{1}{3}V_c\overline{\backslash 120^\circ} = V_a^+ - V_a^+\underline{/120^\circ} \tag{93}$$

which, upon contraction, may be written

$$\sqrt{3}V_a\overline{\backslash 30^\circ} + \sqrt{3}V_b\underline{/90^\circ} - \sqrt{3}V_c\underline{/30^\circ} = 3\sqrt{3}V_a^+\overline{\backslash 30^\circ} \tag{94}$$

Solving for V_a^+ gives

$$V_a^+ = \tfrac{1}{3}(V_a + V_b\underline{/120^\circ} + V_c\overline{\backslash 120^\circ}) \tag{95}$$

V_a^- may now be determined by substituting equation (95) in equation (89). Thus,

$$\tfrac{2}{3}V_a - \tfrac{1}{3}V_a - \tfrac{1}{3}V_b - \tfrac{1}{3}V_b\underline{/120^\circ} - \tfrac{1}{3}V_c - \tfrac{1}{3}V_c\overline{\backslash 120^\circ} = V_a^- \tag{96}$$

giving

$$V_a^- = \tfrac{1}{3}(V_a + V_b\overline{\backslash 120^\circ} + V_c\underline{/120^\circ}) \tag{97}$$

Equations (95) and (97) give the positive- and negative-sequence voltages in terms of the three unbalanced voltages. If the latter add to zero or if the zero-sequence component be subtracted from the original unbalanced voltages so that the remaining vectors form a closed triangle, the positive- and negative-sequence components may be expressed in terms of two of the

voltages only.[1] Indicating the unbalanced voltages, which add to zero by primes, the following equation may be written:

$$V'_a = V_a - V_0 = V_a^+\underline{/0} + V_a^-\underline{/0} \tag{98}$$

$$V'_b = V_b - V_0 = V_a^+\overline{\backslash 120^\circ} + V_a^-\underline{/120^\circ} \tag{99}$$

$$V'_c = V_c - V_0 = V_a^+\underline{/120^\circ} + V_a^-\overline{\backslash 120^\circ} \tag{100}$$

Operating on equation (99) by $\overline{\backslash 120^\circ}$ gives

$$V'_b\overline{\backslash 120^\circ} = V_a^+\underline{/120^\circ} + V_a^-\underline{/0} \tag{101}$$

Subtracting equation (101) from equation (98) eliminates V^- and leaves

$$V'_a - V'_b\overline{\backslash 120^\circ} = V_a^+\underline{/0} - V_a^+\underline{/120^\circ} \tag{102}$$

or

$$V'_a + V'_b\underline{/60^\circ} = \sqrt{3}V_a^+\overline{\backslash 30^\circ} \tag{103}$$

Solving for V_a^+ gives

$$V_a^+ = \frac{1}{\sqrt{3}}\left(V'_a + V'_b\underline{/60^\circ}\right)\underline{/30^\circ} \tag{104}$$

Substituting this value of V_a^+ in equation (98) gives

$$V_a^- = V'_a - \frac{V'_a}{\sqrt{3}}\underline{/30^\circ} - \frac{V'_b}{\sqrt{3}}\underline{/90^\circ} \tag{105}$$

Hence, upon contraction,

$$V_a^- = \frac{1}{\sqrt{3}}\left(V'_a + V'_b\overline{\backslash 60^\circ}\right)\overline{\backslash 30^\circ} \tag{106}$$

Determination of the positive- and negative-sequence components by means of equations (104) and (106) is particularly convenient when no zero-sequence components are present. In this case, the original voltages V_a and V_b are used in the formulas, since obviously $V_a = V'_a$ and $V_b = V'_b$.

The components of a three-phase system of unbalanced currents may be determined exactly as discussed for the voltages. The formulas for voltages as well as currents will be collected below for rapid reference.

First Method:

Voltage components:

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c) \tag{107}$$

$$V_a^+ = \tfrac{1}{3}(V_a + V_b\underline{/120^\circ} + V_c\overline{\backslash 120^\circ}) \tag{108}$$

$$V_a^- = \tfrac{1}{3}(V_a + V_b\overline{\backslash 120^\circ} + V_c\underline{/120^\circ}) \tag{109}$$

[1] Lyon, W. V., "Unbalanced Three-phase Circuits," *Elec. World*, p. 1304, 1920.

Current components:

$$I_0 = \tfrac{1}{3}(I_a + I_b + I_c) \tag{110}$$

$$I_a^+ = \tfrac{1}{3}(I_a + I_b\underline{/120^\circ} + I_c\overline{\backslash 120^\circ}) \tag{111}$$

$$I_a^- = \tfrac{1}{3}(I_a + I_b\overline{\backslash 120^\circ} + I_c\underline{/120^\circ}) \tag{112}$$

Second Method:

Voltage components:

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c) \tag{113}$$

$$V_a' = V_a - V_0 \tag{114}$$

$$V_b' = V_b - V_0 \tag{115}$$

$$V_a^+ = \frac{1}{\sqrt{3}}(V_a' + V_b'\underline{/60^\circ})\underline{/30^\circ} \tag{116}$$

$$V_a^- = \frac{1}{\sqrt{3}}(V_a' + V_b'\overline{\backslash 60^\circ})\overline{\backslash 30^\circ} \tag{117}$$

Current components:

$$I_0 = \tfrac{1}{3}(I_a + I_b + I_c) \tag{118}$$

$$I_a' = I_a - I_0 \tag{119}$$

$$I_b' = I_b - I_0 \tag{120}$$

$$I_a^+ = \frac{1}{\sqrt{3}}(I_a' + I_b'\underline{/60^\circ})\underline{/30^\circ} \tag{121}$$

$$I_a^- = \frac{1}{\sqrt{3}}(I_a' + I_b'\overline{\backslash 60^\circ})\overline{\backslash 30^\circ} \tag{122}$$

EXAMPLE 2

Statement of Problem

In a 230-volt, three-phase, Y-connected circuit, one of the permanently connected line voltmeters reads low, thus indicating the presence of unbalance. In order to check up on the conditions, the three-line voltages and two of the voltages to neutral are accurately recorded as follows:

$V_{ab} = 200$ volts	$V_{an} = 133$ volts
$V_{bc} = 230$ volts	$V_{bn} = 133$ volts
$V_{ca} = 230$ volts	

1. What is the third voltage to neutral, V_{cn}?
2. Determine the symmetrical phase components of the voltages to neutral.
3. Determine the symmetrical phase components of the line voltages.

Solution

Figure 44 is a vector diagram of the line and phase voltages.

1. The third voltage to neutral is

$$V_{cn} = \sqrt{230^2 - 100^2} - \sqrt{133^2 - 100^2} = 207.1 - 87.6 = 119.5 \text{ volts}$$

2. $\cos\alpha = {}^{100}\!/_{133} = 0.752 \qquad \alpha = 41.2$ deg.

Using V_{ab} as standard phase, the vector expressions for the voltages to neutral are

$$V_{an} = 133\overline{\backslash 41^\circ.2} = 100.0 - j87.6 \text{ volts}$$
$$V_{bn} = 133\overline{\backslash 138^\circ.8} = -100.0 - j87.6 \text{ volts}$$
$$V_{cn} = 119.5\underline{/90^\circ} = j119.5 \text{ volts}$$

The symmetrical components of these voltages become

$$V_o = \tfrac{1}{3}(V_{an} + V_{bn} + V_{cn})$$
$$= \tfrac{1}{3}(-j175.2 + j119.5) = -j18.6 = 18.6\overline{\backslash 90^\circ} \text{ volts}$$
$$V_a^+ = \tfrac{1}{3}(V_{an} + V_{bn}\underline{/120^\circ} + V_{cn}\overline{\backslash 120^\circ})$$
$$= \tfrac{1}{3}(133\overline{\backslash 41^\circ.2} + 133\overline{\backslash 18^\circ.8} + 119.5\overline{\backslash 30^\circ})$$
$$= \tfrac{1}{3}(100 - j87.6 + 125.9 - j42.8 + 103.6 - j59.8)$$
$$= \tfrac{1}{3}(329.5 - j190.2) = 109.8 - j63.4 = 126.7\overline{\backslash 30^\circ.0} \text{ volts}$$
$$V_a^- = \tfrac{1}{3}(V_{an} + V_{bn}\overline{\backslash 120^\circ} + V_{cn}\underline{/120^\circ})$$
$$= \tfrac{1}{3}(133\overline{\backslash 41^\circ.2} + 133\underline{/101^\circ.2} + 119.5\overline{\backslash 150^\circ})$$
$$= \tfrac{1}{3}(100 - j87.6 - 25.8 + j130.6 - 103.6 - j59.8)$$
$$= \tfrac{1}{3}(-29.4 - j16.8) = -9.8 - j5.6 = 11.3\overline{\backslash 150^\circ.2} \text{ volts}$$

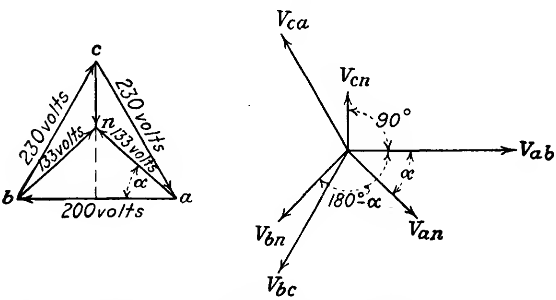

Fig. 44.—Vector diagrams of line voltages and voltages to neutral (Example 2).

Check

$$V_0 + V_a^+ + V_a^- = -j18.6 + 109.8 - j63.4 - 9.8 - j5.6$$
$$= 100.0 - j87.6 = V_{an}$$

3. The line voltages contain no zero-sequence components. The positive- and negative-sequence components of the line voltage V_{ab} are

$$V_{ab}^+ = \sqrt{3}\, V_{an}^+ \underline{/30^\circ} = \sqrt{3} \times 126.7\,\overline{\backslash 30^\circ}\,\underline{/30^\circ} = 219.6\underline{/0} \text{ volts}$$
$$V_{ab}^- = \sqrt{3}\, V_{an}^- \overline{\backslash 30^\circ} = \sqrt{3} \times 11.3\,\overline{\backslash 150^\circ.2}\,\overline{\backslash 30^\circ} = 19.6\,\overline{\backslash 180^\circ.2} \text{ volts}$$

Check

$$V_{ab}^+ + V_{ab}^- = 219.6 - 19.6 + j0.07 = 200 + j0.07 \cong 200 = V_{ab}$$

Graphical Determination of the Components in a Three-phase System.—When the desired degree of precision is not too great, the symmetrical phase components of an unbalanced system of three-phase vectors may be found graphically. The graphical methods are based directly on the previously developed formulas;

i.e., the operations indicated by the equations are simply carried out graphically by drawing the vectors involved to a suitable scale.

In Fig. 45, the vectors *a* represent the unbalanced voltages. Addition, as in *b*, immediately gives the zero-sequence compo-

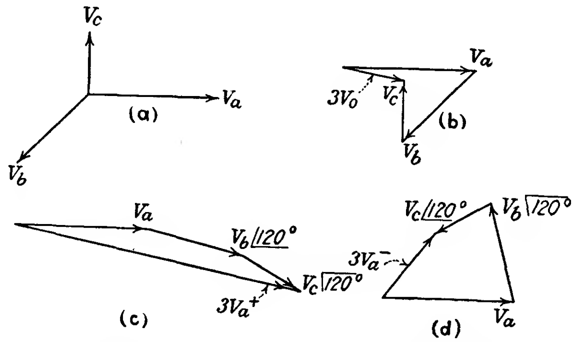

FIG. 45.—Graphical determination of positive-, negative-, and zero-sequence voltage components in a three-phase system.

nents in magnitude and phase. By rotating the vector V_b through 120 deg. in a positive direction and V_c through 120 deg. in a negative direction and adding them to the vector V_a, as indicated in *c*, the positive-sequence component of phase *a* is determined. Finally, by rotating the vector V_b through 120 deg.

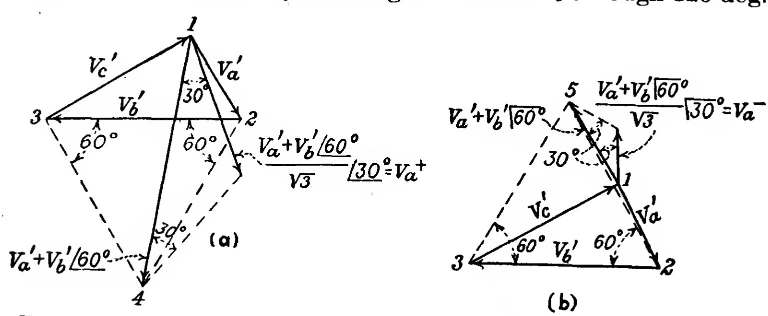

FIG. 46.—Graphical determination of positive- and negative-sequence voltage components in a three-phase system when the zero-sequence component is either absent or subtracted from the original unbalanced voltages.

in a negative direction and the vector V_c through 120 deg. in a positive direction and again adding them to the vector V_a, as shown in *d*, the negative-sequence component of phase *a* is obtained. It will be noted that the procedure followed is in accordance with equations (107), (108), and (109).

Figure 46 illustrates a graphical solution applicable to unbalanced vectors whose sum is zero. It is based on equations (116) and (117). The former calls for rotation of the vector V_b through 60 deg. in a positive direction. This operation is readily carried out by constructing an equilateral triangle on this vector, as shown in *a*. Evidently, the line 1 to 4 is then equal to $V'_a + V'_b\underline{/60°}$. This quantity is now to be divided by $\sqrt{3}$ and rotated through 30 deg. in a positive direction. Construction of an isosceles triangle with 1 to 4 as the base line by laying off the two 30-deg. angles immediately accomplishes this end and locates V_a^+ in magnitude and phase.

In order to obtain the negative-sequence component, the construction is carried out in accordance with equation (117). The equilateral triangle on V'_b is now constructed as indicated in *b*. The vector V'_b is thus rotated 60 deg. in a negative direction. The line 1 to 5 is equal to $V'_a + V'_b\overline{\backslash 60°}$, and by constructing an isosceles triangle with this line as base by laying off two 30-deg. angles V_a^- is obtained in magnitude and phase, as indicated.

As seen, this graphical scheme is extremely simple. It is particularly suitable when only the magnitudes of the components are required. These can be obtained by merely constructing the two equilateral triangles and scaling off the lines 1–4 and 1–5. These lines divided by $\sqrt{3}$ give the magnitudes of the positive- and negative-sequence components, respectively.

Analytical Determination of the Components in Two-phase and Four-phase Systems.—As already stated, there is no necessity of considering any zero-sequence components of voltage and current in a two-phase system. The two voltages, for instance, can always be resolved into positive and negative-sequence components, as indicated by equations (69) and (70). Operating on equation (70) by $\underline{/90°}$ gives

$$V_b\underline{/90°} = V_a^+\underline{/0} + V_a^-\underline{/180°} \tag{123}$$

Adding this to equation (69) eliminates V_a^-, and the following solution for V_a^+ is obtained

$$V_a^+ = \tfrac{1}{2}(V_a + V_b\underline{/90°}) \tag{124}$$

Substituting this value in equation (69) gives for the negative-sequence components

$$V_a^- = \tfrac{1}{2}(V_a + V_b\overline{\backslash 90°}) \tag{125}$$

Similar expressions may be written for the current components.

In a four-phase (quarter-phase) system, zero-sequence components may be present in the voltages to neutral as well as in the current. Hence, in order to cover the general case, these components must be considered. The zero-sequence voltage is given by

$$V_0 = \tfrac{1}{4}(V_a + V_b + V_c + V_d) \tag{126}$$

Substituting this in equations (77) to (80) inclusive gives

$$\tfrac{3}{4}V_a - \tfrac{1}{4}V_b - \tfrac{1}{4}V_c - \tfrac{1}{4}V_d = V_a^+ \underline{/0} + V_a^- \underline{/0} + V^* \underline{/0} \tag{127}$$

$$-\tfrac{1}{4}V_a + \tfrac{3}{4}V_b - \tfrac{1}{4}V_c - \tfrac{1}{4}V_d = V_a^+ \overline{\backslash 90^\circ} + V_a^- \underline{/180^\circ} + V_a^* \underline{/90^\circ} \tag{128}$$

$$-\tfrac{1}{4}V_a - \tfrac{1}{4}V_b + \tfrac{3}{4}V_c - \tfrac{1}{4}V_d = V_a^+ \underline{/180^\circ} + V_a^- \underline{/0} + V_a^* \underline{/180^\circ} \tag{129}$$

$$-\tfrac{1}{4}V_a - \tfrac{1}{4}V_b - \tfrac{1}{4}V_c + \tfrac{3}{4}V_d = V_a^+ \underline{/90^\circ} + V_a^- \underline{/180^\circ} + V^* \overline{\backslash 90^\circ} \tag{130}$$

By adding equations (127) and (128) and equations (128) and (129), the following equations are obtained:

$$\tfrac{1}{2}V_a + \tfrac{1}{2}V_b - \tfrac{1}{2}V_c - \tfrac{1}{2}V_d = \sqrt{2}\, V_a^+ \overline{\backslash 45^\circ} + \sqrt{2}\, V_a^* \underline{/45^\circ} \tag{131}$$

$$-\tfrac{1}{2}V_a + \tfrac{1}{2}V_b + \tfrac{1}{2}V_c - \tfrac{1}{2}V_d = -\sqrt{2}\, V_a^+ \underline{/45^\circ} - \sqrt{2}\, V_a^* \overline{\backslash 45^\circ} \tag{132}$$

Operating on equation (132) by $\overline{\backslash 90^\circ}$ and subtracting from equation (131) eliminates V_a^* and gives

$$\sqrt{2}\, V_a \overline{\backslash 45^\circ} + \sqrt{2}\, V_b \underline{/45^\circ} - \sqrt{2}\, V_c \overline{\backslash 45^\circ} - \sqrt{2}\, V_d \underline{/45^\circ} = 4\sqrt{2}\, V_a^+ \overline{\backslash 45^\circ} \tag{133}$$

from which

$$V_a^+ = \tfrac{1}{4}(V_a + V_b \underline{/90^\circ} + V_c \underline{/180^\circ} + V_d \overline{\backslash 90^\circ}) \tag{134}$$

Addition of equations (127) and (129) eliminates both V_a^+ and V_a^*, giving

$$\tfrac{1}{2}V_a - \tfrac{1}{2}V_b + \tfrac{1}{2}V_c - \tfrac{1}{2}V_d = 2V_a^- \underline{/0} \tag{135}$$

Hence, V_a^- becomes

$$V_a^- = \tfrac{1}{4}(V_a + V_b \underline{/180^\circ} + V_c + V_d \underline{/180^\circ}) \tag{136}$$

Finally, by substituting equation (134) in either equation (131) or equation (132), V_a^* is obtained, as

$$V_a^* = \tfrac{1}{4}(V_a + V_b \overline{\backslash 90^\circ} + V_c \underline{/180^\circ} + V_d \underline{/90^\circ}) \tag{137}$$

Graphical Deterimination of the Components in Two-phase and Four-phase Systems.—By basing the construction on the established formulas, graphical solutions may be obtained for the sym-

metrical components of unbalanced two-phase and four-phase systems.

Figure 47 shows such a graphical solution for the positive- and negative-sequence voltages of a two-phase system. The process

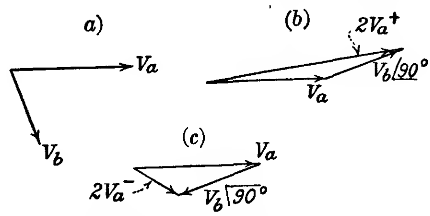

FIG. 47.—Graphical determination of positive- and negative-sequence voltage components in a two-phase system.

used is in accordance with equations (124) and (125). The original unbalanced vectors are shown in diagram *a*. The vector V_b is rotated through 90 deg. in a positive direction and added to

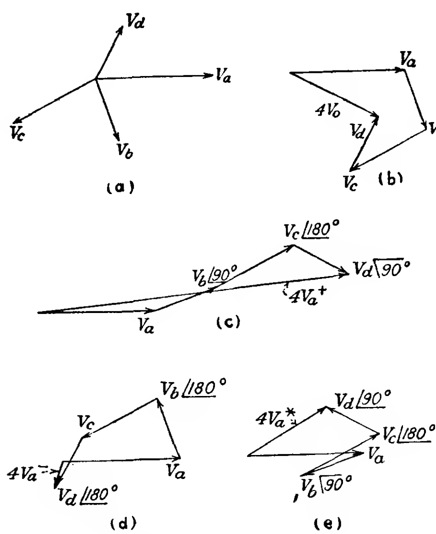

FIG. 48.—Graphical determination of positive-, negative-, and zero-sequence voltage components in a four-phase system.

V_a, as indicated in *b*. Thus, V_a^+ is obtained in magnitude and phase. In diagram *c*, the vector V_b is rotated through 90 deg. in a negative direction and added to V_a, giving the negative-sequence voltage V_a^-.

Figure 48 shows the construction for a four-phase system, based on equations (126), (134), (136), and (137). Diagram *a* shows the original unbalanced four-phase vectors. Diagram *b* gives the sum of the vectors and thus determines the zero-sequence voltage V_0. In *c*, the vector V_b is rotated through 90 deg. in a positive direction, V_c through 180 deg., and V_d through 90 deg. in a negative direction. The sum of the vectors when laid off in this manner is four times V_a^+. Diagram *d* gives V_a^- by adding the vectors after having rotated V_b and V_d through 180 deg. In diagram *e*, V_b is rotated through 90 deg. in a negative direction, V_c through 180 deg., and V_d through 90 deg. in a positive direction. This construction determines V_a^*.

Power in Unbalanced Circuits.—The power per phase of an unbalanced circuit evidently depends on the voltage, current, and power-factor angle of that phase. In general, the power developed or utilized in the various phases will not be the same when unbalance is present.

These statements are perfectly general and hold for any number of phases. The discussion below, however, will be more specifically confined to the unbalanced three-phase circuit. The power in phases *a*, *b*, and *c* are given by

$$P_a = V_a I_a \cos \theta_a \tag{140}$$

$$P_b = V_b I_b \cos \theta_b \tag{141}$$

$$P_c = V_c I_c \cos \theta_c \tag{142}$$

The voltages and currents in these equations, as well as in the subsequent equations for power, are scalar quantities and not vectors. When the voltages and currents are resolved into positive-, negative-, and zero-sequence components, the power in phase *a* may be written

$$\begin{aligned} P_a = {} & V_a^+ I_a^+ \cos \theta_{I_a^+}^{V_a^+} + V_a^- I_a^- \cos \theta_{I_a^-}^{V_a^-} + V_0 I_0 \cos \theta_{I_0}^{V_0} + \\ & V_a^+ I_a^- \cos \theta_{I_a^-}^{V_a^+} + V_a^+ I_0 \cos \theta_{I_0}^{V_a^+} + V_a^- I_a^+ \cos \theta_{I_a^+}^{V_a^-} + \\ & V_a^- I_0 \cos \theta_{I_0}^{V_a^-} + V_0 I_a^+ \cos \theta_{I_a^+}^{V_0} + V_0 I_a^- \cos \theta_{I_a^-}^{V_0} \end{aligned} \tag{143}$$

Similar expressions may also be written for phases *b* and *c*. From these equations, the actual power *per phase* may be calculated.

The total power in a three-phase system equals the sum of the phase powers. Thus,

$$P = P_a + P_b + P_c \tag{144}$$

The total power is readily evaluated in terms of the component voltages, currents, and power-factor angles by adding equation (143) and the corresponding equations for the two other phases. In performing this addition, it will be found that the sums of the terms involving unlike components are zero. Since the magnitude of like components is the same in all three phases, the subscripts referring to phase may be dropped and the expression for the total power written as

$$\begin{aligned} P &= 3(V^+I^+ \cos \theta^+ + V^-I^- \cos \theta^- + V_0I_0 \cos \theta_0) \\ &= 3(P^+ + P^- + P_0) \end{aligned} \tag{145}$$

Hence, the total power in an unbalanced three-phase circuit is equal to the sum of the powers due to the positive-sequence, negative-sequence, and zero-sequence components.

Quite often the zero-sequence components are absent, and power is produced by positive- and negative-sequence components only. In such cases, the power in at least one phase will be greater and in at least one phase less than the average power per phase. The effect of the negative-sequence components is thus to transfer power from one phase to another. This ability of the negative-sequence components to transfer power is utilized in the phase balancer.[1]

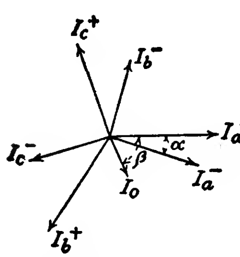

FIG. 49.—Vector diagram of positive-, negative-, and zero-sequence currents in a three-phase system.

Copper Losses in Unbalanced Circuits. The copper loss in any phase of an unbalanced circuit equals the product of the square of the actual current carried by that phase times its resistance. Thus, for a three-phase system,

$$P_{a(Cu)} = I_a^2 r_a = (I_a^+ + I_a^- + I_0)^2 r_a \tag{146}$$

$$P_{b(Cu)} = I_b^2 r_b = (I_b^+ + I_b^- + I_0)^2 r_b \tag{147}$$

$$P_{c(Cu)} = I_c^2 r_c = (I_c^+ + I_c^- + I_0)^2 r_c \tag{148}$$

If there is a neutral connection carrying current, the copper loss in this is given by

$$P_{n(Cu)} = I_n^2 r_n = (I_a + I_b + I_c)^2 r_n = 9I_0^2 r_n \tag{149}$$

[1] TORCHIO, PHILIP, "Supply of Single-phase Loads from Central Stations," *Trans. A.I.E.E.*, p.1293, 1916.

ALEXANDERSON, E. F. W., and G. H. HILL, "Single-phase Power Production," *Trans. A.I.E.E.*, p. 1315, 1916.

GILMAN, R. E., and C. L. FORTESCUE, "Single-phase Power Service from Central Stations," *Trans. A.I.E.E.*, p. 1329, 1916.

Assume that Fig. 49 is the vector diagram of component currents. Making use of this diagram, equations (146), (147), and (148) for the copper losses per phase may be written

$$
\begin{aligned}
P_{a(Cu)} &= [(I_a^+ + I_a^- \cos\alpha + I_0\cos\beta)^2 + (I_a^- \sin\alpha + I_0 \sin\beta)^2]r_a \\
&= [(I_a^+)^2 + (I_a^-)^2 + I_0^2]r_a + \\
&\quad 2[I_a^+ I_a^- \cos\alpha + I_a^+ I_0 \cos\beta + I_a^- I_0 \cos\alpha\cos\beta + \\
&\qquad I_a^- I_0 \sin\alpha \sin\beta]r_a \qquad (150)
\end{aligned}
$$

$$
\begin{aligned}
P_{b(Cu)} &= [\{I_b^+ + I_b^- \cos(240^\circ - \alpha) + I_0 \cos(120^\circ - \beta)\}^2 + \\
&\quad \{I_b^- \sin(240^\circ - \alpha) + I_0 \sin(120^\circ - \beta)\}^2]r_b \\
&= [(I_b^+)^2 + (I_b^-)^2 + I_0^2]r_b + \\
&\quad 2[I_b^+ I_b^- \cos(240^\circ - \alpha) + I_b^+ I_0 \cos(120^\circ - \beta) + \\
&\quad I_b^- I_0 \cos(240^\circ - \alpha)\cos(120^\circ - \beta) + \\
&\quad I_b^- I_0 \sin(240^\circ - \alpha)\sin(120^\circ - \beta)]r_b \qquad (151)
\end{aligned}
$$

$$
\begin{aligned}
P_{c(Cu)} &= [\{I_c^+ + I_c^- \cos(120^\circ - \alpha) + I_0 \cos(240^\circ - \beta)\}^2 + \\
&\quad \{I_c^- \sin(120^\circ - \alpha) + I_0 \sin(240^\circ - \beta)\}^2]r_c \\
&= [(I_c^+)^2 + (I_c^-)^2 + I_0^2]r_c + \\
&\quad 2[(I_c^+ I_c^- \cos(120^\circ - \alpha) + I_c^+ I_0 \cos(240^\circ - \beta) + \\
&\quad I_c^- I_0 \cos(120^\circ - \alpha)\cos(240^\circ - \beta) + \\
&\quad I_c^- I_0 \sin(120^\circ - \alpha)\sin(240^\circ - \beta)]r_c \qquad (152)
\end{aligned}
$$

The total copper loss in the three phases may be obtained by adding equations (150), (151), and (152). Dropping subscripts referring to phase and assuming that the phase resistances are identical, the total copper loss (exclusive of possible loss in the neutral) becomes

$$
\begin{aligned}
P_{(Cu)} &= 3[(I^+)^2 + (I^-)^2 + I_0^2]r + \\
&\quad 2\{I^+I^-[\cos\alpha + \cos(120^\circ - \alpha) + \cos(240^\circ - \alpha)] + \\
&\quad I^+I_0[\cos\beta + \cos(120^\circ - \beta) + \cos(240^\circ - \beta)] + \\
&\quad I^-I_0[\cos\alpha\cos\beta + \cos(240^\circ - \alpha)\cos(120^\circ - \beta) + \\
&\qquad \cos(120^\circ - \alpha)\cos(240^\circ - \beta)] + \\
&\quad I^-I_0[\sin\alpha\sin\beta + \sin(240^\circ - \alpha)\sin(120^\circ - \beta) + \\
&\qquad \sin(120^\circ - \alpha)\sin(240^\circ - \beta)]\}r \qquad (153)
\end{aligned}
$$

The sum of the trigonometric functions in each of the brackets is zero. Hence,

$$
P_{(Cu)} = 3[(I^+)^2 + (I^-)^2 + (I_0)^2]r \qquad (154)
$$

The total copper loss in the three phases is thus equal to the sum of the copper losses due to the positive-, negative-, and zero-sequence currents considered separately.

Above, it has tacitly been assumed that the resistance was the *same for the several components of current.* This is true for lines and transformers but is not, in general, true for rotating machines.

When these resistances are different, the total copper loss is given by

$$P_{(Cu)} = 3[(I^+)^2r^+ + (I^-)^2r^- + I_0^2r_0] \tag{155}$$

Of course, equation (155) also assumes that the component resistances of the three phases are identical.

Component Voltages in a Three-phase System in Terms of Impedance Drops. Equivalent Positive-, Negative-, and Zero-sequence Impedance of an Unbalanced Load.—Consider a circuit, as shown in Fig. 50, consisting of three unbalanced Y-connected impedances whose values are the same for the component systems of current. In other words, the positive-, negative-, and zero-sequence impedance of each phase are identical.

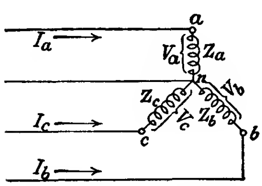

FIG. 50.—Y-connected unbalanced impedances. Neutral return.

V_a, V_b, and V_c are the voltage drops across the three impedances, respectively. They may be written

$$V_a = I_aZ_a \tag{156}$$
$$V_b = I_bZ_b \tag{157}$$
$$V_c = I_cZ_c \tag{158}$$

From previous derivations, the component voltages in terms of the actual voltages are given by

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c) \tag{159}$$
$$V_a^+ = \tfrac{1}{3}(V_a + V_b\underline{/120^\circ} + V_c\overline{\backslash 120^\circ}) \tag{160}$$
$$V_a^- = \tfrac{1}{3}(V_a + V_b\overline{\backslash 120^\circ} + V_c\underline{/120^\circ}) \tag{161}$$

Substituting equations (156), (157), and (158) in equation (159) gives the zero-sequence voltage

$$\begin{aligned} V_0 &= \tfrac{1}{3}(I_aZ_a + Z_bI_b + I_cZ_c) \\ &= \tfrac{1}{3}[(I_a^+ + I_a^- + I_0)Z_a + (I_a^+\overline{\backslash 120^\circ} + I_a^-\underline{/120^\circ} + I_0)Z_b + \\ &\qquad (I_a^+\underline{/120^\circ} + I_a^-\overline{\backslash 120^\circ} + I_0)Z_c] \\ &= \tfrac{1}{3}[I_a^+(Z_a + Z_b\overline{\backslash 120^\circ} + Z_c\underline{/120^\circ}) + I_a^-(Z_a + Z_b\underline{/120^\circ} + \\ &\qquad Z_c\overline{\backslash 120^\circ}) + I_0(Z_a + Z_b + Z_c)] \\ &= I_a^+Z_- + I_a^-Z_+ + I_0Z_0 \end{aligned} \tag{162}$$

where

$$Z_+ = \tfrac{1}{3}(Z_a + Z_b\underline{/120^\circ} + Z_c\overline{\backslash 120^\circ}) \tag{163}$$
$$Z_- = \tfrac{1}{3}(Z_a + Z_b\overline{\backslash 120^\circ} + Z_c\underline{/120^\circ}) \tag{164}$$
$$Z_0 = \tfrac{1}{3}(Z_a + Z_b + Z_c) \tag{165}$$

By substituting equations (156), (157), and (158) in equations (160) and (161), the positive- and negative-sequence voltage drops in phase a may be written

$$V_a^+ = I_a^+ Z_0 + I_a^- Z_- + I_0 Z_+ \tag{166}$$

$$V_a^- = I_a^+ Z_+ + I_a^- Z_0 + I_0 Z_- \tag{167}$$

Equations (162), (166), and (167) express the zero-, positive-, and negative-sequence voltage drops in terms of the component currents and a combination of the impedances. The impedance Z_+, Z_-, and Z_0 may be termed the *equivalent* positive-, negative-, and zero-sequence impedances, respectively, of the unbalanced three-phase circuit. As the above equations indicate—and this should be carefully noted and appreciated—these equivalent impedances are not positive-, negative-, and zero-sequence impedances in the ordinary sense, *i.e.*, in the sense that each should be used exclusively with the current of the corresponding sequence. They are named equivalent positive-, negative-, and zero-sequence impedances merely because they are evaluated from the unbalanced impedances (equations (163), (164), and (165)) by a process exactly similar to the one used in determining the symmetrical components of a system of unbalanced voltages or currents (equations (159), (160), and (161)). It is seen that, in general, all three current components contribute to each component drop. If the circuit is symmetrical, however, and $Z_a = Z_b = Z_c = Z$, the impedances Z_+ and Z_- are both zero, while $Z_0 = Z$. In such a circuit, therefore, the positive-sequence drop is caused by a positive-sequence current only, the negative-sequence drop by a negative-sequence current only, and the zero-sequence drop by a zero-sequence current only.

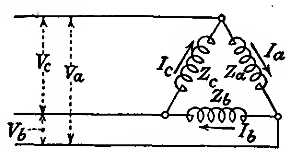

FIG. 51.—Δ-connected unbalanced impedances.

When the Y-connected circuit has no neutral connection, the zero-sequence current is suppressed. If, at the same time, the circuit is symmetrical, the zero-sequence voltage must always be zero, independent of whether or not the line voltages are balanced.

An unsymmetrical Δ-connected circuit (Fig. 51) may be treated exactly as the Y-connected circuit, and the impedances combined in the same manner. With the Δ-connection, the zero-sequence voltage is always zero.

Hence, equation (162), giving the zero-sequence voltage in terms of component impedance drops, can be equated to zero,

and the following expression for the zero-sequence current obtained:

$$I_0 = -\frac{I_a^+ Z_- + I_a^- Z_+}{Z_0} \tag{168}$$

Equations (166) and (167) still hold for the positive- and negative- sequence voltages. Substituting, however, the value of I_0 from equation (168) in equations (166) and (167), the positive- and negative-sequence voltages may be written in terms of impedance drops due to positive- and negative-sequence currents only. Thus,

$$V_a^+ = I_a^+\left(Z_0 - \frac{Z_+ Z_-}{Z_0}\right) + I_a^-\left(Z_- - \frac{Z_+^2}{Z_0}\right) \tag{169}$$

$$V_a^- = I_a^+\left(Z_+ - \frac{Z_-^2}{Z_0}\right) + I_a^-\left(Z_0 - \frac{Z_+ Z_-}{Z_0}\right) \tag{170}$$

If the Δ-circuit is symmetrical, the zero-sequence current must always be zero, independent of whether or not the impressed voltages are balanced. This is readily seen from equation (168), when it is borne in mind that both Z_+ and Z_- become zero for a symmetrical circuit.

Degree of Unbalance.—When the voltages and currents in a polyphase circuit are unbalanced, it is sometimes convenient to express, numerically, the "degree of unbalance" which may be said to exist in the circuit. A conventional expression for degree of unbalance, therefore, should be fixed by definition.

When the conditions in a circuit are completely balanced, only positive-sequence voltages and currents are present. Hence, it seems logical in an unbalanced circuit to use the numerical ratio of the negative-sequence and zero-sequence components, respectively, to the positive-sequence component as an indication of the degree of unbalance. These ratios designating the degree of unbalance in a circuit may also be termed "unbalance factors."[1] In general, the degrees of voltage and current unbalance will be different and should be calculated separately.

At any point in a circuit, the degree of voltage unbalance is given by the degree of negative-sequence unbalance and the degree of zero-sequence unbalance; thus,

$$(\text{Degree of unbalance})^- = \frac{V^-}{V^+} \tag{171}$$

$$(\text{Degree of unbalance})^0 = \frac{V_0}{V^+} \tag{172}$$

[1] See, for instance, J. SLEPIAN, "Induction Motors on Unbalanced Circuits," *Elec. World*, p. 313, 1920.

Similarly, the degree of current unbalance at any point in a circuit is given by

$$(\text{Degree of unbalance})^{-} = \frac{I^{-}}{I^{+}} \qquad (173)$$

$$(\text{Degree of unbalance})^{0} = \frac{I_0}{I^{+}} \qquad (174)$$

In order to specify the degree of voltage or current unbalance completely, two values must be given, one for the negative-sequence and one for the zero-sequence degree of unbalance. There is, apparently, no way of combining these into one readily interpretable value. Only when either the negative-sequence or the zero-sequence system is absent will one figure suffice.

The degree of unbalance may have any numerical value *from zero to infinity.* This may seem rather astonishing at first thought, and, as a matter of fact, abnormally high degrees of unbalance are seldom met with. It is probably safe to say that 100 per cent unbalance is the maximum which is likely to occur in power circuits, and even this degree of unbalance will, as a rule, be encountered only under abnormal operating conditions. It is possible in such cases, however, for 100 per cent negative sequence unbalance and 100 per cent zero-sequence unbalance to occur simultaneously.

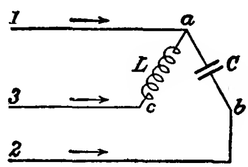

FIG. 52.—Pure inductance and capacitance V-connected. The values of inductance and capacitance are such that the two branches have identical numerical reactance.

In order to illustrate the possibility of infinite unbalance, which obviously represents the most extreme case, consider the circuit shown in Fig. 52. Balanced three-phase voltages are impressed on a pure inductance and capacitance connected in open delta. Evidently, this circuit is only approximately realizable, since any inductance would have *some* resistance and any condenser *some* leakance. If the inductance and capacitance be given such values that they offer the same numerical reactance, the magnitude of the currents flowing in the two branches of the open delta will be the same.

The vector diagram of voltages and currents in this circuit is given in Fig. 53. Inspection of the vectors representing line currents reveals the fact that they *form a balanced system whose phase order is opposite to that of the voltages.* If the phase order of the voltages, therefore, is considered positive, *the currents are all*

negative-sequence currents. They contain no positive-sequence or zero-sequence components. The degree of (negative-sequence) current unbalance will consequently be

$$(\text{Degree of unbalance})^- = \frac{I^-}{I^+} = \frac{I^-}{0} = \infty$$

Figure 54 shows a circuit in which a pure inductance and capacitance are connected line to neutral in two of the phases of a

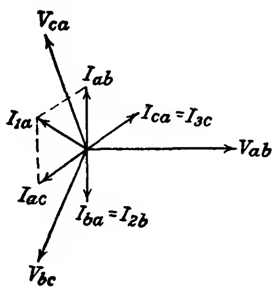

Fig. 53.

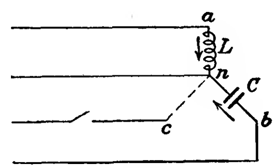

Fig. 54.

Fig. 53.—Vector diagram of voltages and currents in the circuit in Fig. 52.
Fig. 54.—Diagram of a circuit in which a pure inductance and a pure capacitance are connected line to neutral. The values of inductance and capacitance are such that the two branches have the same numerical reactance.

three phase circuit. The third phase is open. The voltages to neutral are strictly balanced, and it is assumed that the inductive reactance and the capacitive reactance are numerically equal, so that equal currents flow.

The vector diagram for this circuit is then given in Fig. 55, showing the relative phase relation of the phase voltages and line currents. Considering the voltage V_a as standard phase, the component currents are readily determined by equations (110), (111), and (112) and become

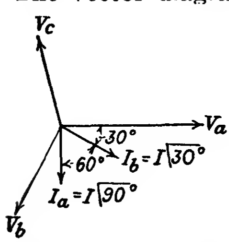

Fig. 55.—Vector diagram of voltages and currents in the circuit in Fig. 54.

$$I_0 = \frac{1}{3}(I\,\overline{\backslash 90^\circ} + I\,\overline{\backslash 30^\circ}) = \frac{I}{\sqrt{3}}\,\overline{\backslash 60^\circ}$$

$$I_a^+ = \tfrac{1}{3}(I\,\overline{\backslash 90^\circ} + I\,\underline{/90^\circ}) = 0$$

$$I_a^- = \frac{1}{3}(I\,\overline{\backslash 90^\circ} + I\,\overline{\backslash 150^\circ}) = \frac{I}{\sqrt{3}}\,\overline{\backslash 120^\circ}$$

As seen, the currents contain *no positive-sequence component.* Both the negative-sequence and the zero-sequence degree of current unbalance, therefore, are infinite. Hence,

$$(\text{Degree of unbalance})^{-} = \frac{I^{-}}{I^{+}} = \frac{I^{-}}{0} = \infty$$

$$(\text{Degree of unbalance})^{0} = \frac{I^{0}}{I^{+}} = \frac{I^{0}}{0} = \infty$$

Next, a few examples will be given to illustrate cases where 100 per cent voltage or current unbalance exists. Consider first the Y-connected three-phase circuit shown in Fig. 56. The line voltages are balanced. It is desired to determine the degree of voltage unbalance when phase a is short-circuited.

Taking the line voltage V_{ab} as standard phase, *i.e.*, $V_{ab} = V\underline{/0}$, the voltages to neutral may be written

$$V_a = 0$$

$$V_b = V\underline{/180^\circ}$$

$$V_c = V\underline{/120^\circ}$$

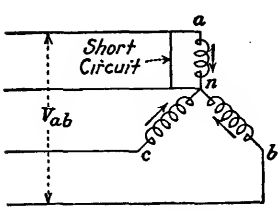

FIG. 56.—Single-phase line-to-neutral short circuit on a three-phase Y-connected circuit.

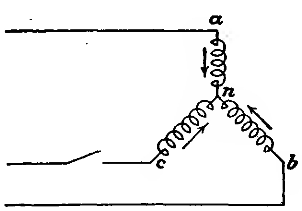

FIG. 57.—Unbalanced three-phase Y-connected circuit without neutral. The phase c is idle.

Since the line is balanced, the magnitude of the positive-sequence phase voltages is $V/\sqrt{3}$ and the negative-sequence components are zero. The zero-sequence component becomes

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c)$$

$$= -\frac{1}{3}(V + V\overline{\backslash 60^\circ}) = -\frac{V}{\sqrt{3}}\overline{\backslash 30^\circ}$$

Hence, the degree of (zero-sequence) voltage unbalance is

$$(\text{Degree of unbalance})^{0} = \frac{V_0}{V^{+}}100 = \frac{V/\sqrt{3}}{V/\sqrt{3}}100 = 100 \text{ per cent}$$

Consider next the three-phase Y-connected circuit without neutral, in Fig. 57. Phase c is opened, thus reducing the circuit

to single phase. It is desired to determine the degree of current and voltage unbalance.

Since there is no neutral connection, no zero-sequence current can flow. The phase currents are

$$I_a = I\underline{/0}$$
$$I_b = I\underline{/180°}$$
$$I_c = 0$$

The positive- and negative-sequence components in phase a become (see equations (111) and (112))

$$I_a^+ = \frac{1}{3}(I\underline{/0} + I\overline{\backslash 60°}) = \frac{I}{\sqrt{3}}\overline{\backslash 30°}$$
$$I_a^- = \frac{1}{3}(I\underline{/0} + I\underline{/60°}) = \frac{I}{\sqrt{3}}\underline{/30°}$$

The degree of (negative-sequence) current unbalance is, consequently,

$$(\text{Degree of unbalance})^- = \frac{I^-}{I^+}100 = \frac{I/\sqrt{3}}{I/\sqrt{3}}100 = 100 \text{ per cent}$$

The degree of voltage unbalance, in this case, depends on whether or not the three load impedances are balanced and on whether or not each branch has identical values of impedance for current of positive and negative sequences. If the impedances are unbalanced, a zero-sequence voltage will appear in addition to the negative-sequence voltage, in spite of the fact that no zero-sequence current flows. If, on the other hand, the impedances are balanced, *no voltage of zero-sequence can possibly appear.* It makes no difference in this connection if the positive- and negative-sequence impedances per phase are the same, or not. This will merely affect the relative magnitudes of the positive- and negative-sequence voltages.

Assuming that each branch offers the same impedance to currents of either sequence but that they are unequal, having the values Z_a, Z_b, and Z_c, respectively, the voltage components may easily be found by applying equations (162), (166), and (167). These give

$$V_0 = I_a^+Z_- + I_a^-Z_+ = \tfrac{1}{3}I_a(Z_a - Z_b)$$
$$V_a^+ = I_a^+Z_0 + I_a^-Z_- = \tfrac{1}{3}I_a(Z_a + Z_b\overline{\backslash 60°})$$
$$V_a^- = I_a^+Z_+ + I_a^-Z_0 = \tfrac{1}{3}I_a(Z_a + Z_b\underline{/60°})$$

The degrees of voltage unbalance thus become

$$(\text{Degree of unbalance})^- = \frac{V^-}{V^+} = \frac{Z_a + Z_b\underline{/60°}}{Z_a + Z_b\overline{\backslash 60°}}$$

$$(\text{Degree of unbalance})^0 = \frac{V_0}{V^+} = \frac{Z_a - Z_b}{Z_a + Z_b\overline{\backslash 60°}}$$

It will be seen that the impedance Z_c of the idle phase, as might be expected, does not affect the results.

When the impedances are balanced ($Z_a = Z_b = Z_c = Z$), the components of voltage to neutral become

$$V_a^+ = \frac{IZ}{\sqrt{3}}\overline{\backslash 30°}$$

$$V_a^- = \frac{IZ}{\sqrt{3}}\underline{/30°}$$

The degree of voltage unbalance is now

$$(\text{Degree of unbalance})^- = \frac{V^-}{V^+}100 = \frac{IZ/\sqrt{3}}{IZ/\sqrt{3}}100 = 100 \text{ per cent}$$

In the given circuit, therefore, it is possible to have 100 per cent voltage and current unbalance occurring at the same time.

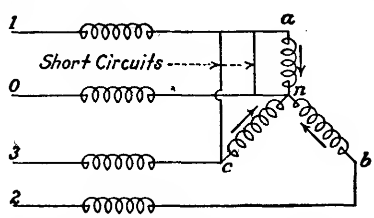

FIG. 58.—Simultaneous line-to-neutral and line-to-line short circuit on a three-phase Y-connected circuit.

Referring to the three-phase circuit shown in Fig. 58, it is assumed that a line-to-neutral short circuit on phase a and a short circuit between lines a and c occur simultaneously. What is the degree of voltage unbalance in this case?

Using the line voltage $V_{ab} = V\underline{/0}$ as standard phase, the voltages to neutral are

$$V_a = 0$$
$$V_b = V_{bc} = -V\underline{/0}$$
$$V_c = 0$$

The symmetrical components of these voltages are then given by

$$V_0 = -\frac{V}{3}\underline{/0}$$

$$V_a^+ = \frac{V}{3}\overline{\backslash 60^\circ}$$

$$V_a^- = \frac{V}{3}\underline{/60^\circ}$$

The degrees of voltage unbalance are, hence,

$$(\text{Degree of unbalance})^- = \frac{V^-}{V^+}100 = \frac{V/3}{V/3}100 = 100 \text{ per cent}$$

$$(\text{Degree of unbalance})^0 = \frac{V_0}{V^+}100 = \frac{V/3}{V/3}100 = 100 \text{ per cent}$$

As seen, both the negative-sequence and zero-sequence degrees of voltage unbalance are 100 per cent in this case. It is possible

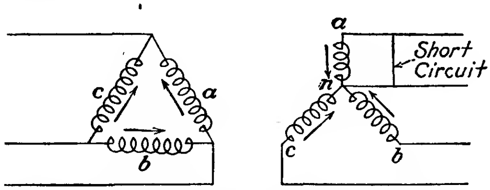

FIG. 59.—Single-phase line-to-neutral short circuit on the secondary side of the Δ-Y-connected bank of transformers.

also to have a similar condition of current unbalance. Consider, for instance, a single-phase line-to-neutral short circuit on the secondary side of a Δ-Y-connected bank of transformers (Fig. 59). The secondary currents may be written

$$I_a = I\underline{/0}$$

$$I_b = 0$$

$$I_c = 0$$

from which

$$I_a^+ = I_a^- = I_0 = \frac{I}{3}\underline{/0}$$

With these components, degrees of current unbalance of 100 per cent would obviously be obtained as follows:

$$(\text{Degree of unbalance})^- = \frac{I^-}{I^+}100 = \frac{I/3}{I/3}100 = 100 \text{ per cent}$$

$$(\text{Degree of unbalance})^0 = \frac{I_0}{I^+}100 = \frac{I/3}{I/3}100 = 100 \text{ per cent}$$

CHAPTER IV

TRANSFORMERS WITH UNBALANCED LOADING

A complete treatment of the general problem of transformers with unbalanced loads, including dissymmetrical short circuits, would require the consideration and analysis of a very great number of cases. Thus, in order to cover all possibilities which may be encountered in present-day practice, both two-phase, three-phase, four-phase, and six-phase systems with all possible transformer connections should be considered. Furthermore, with each connection, solutions should be worked out for all conceivable unsymmetrical loads and short circuits.

Obviously, the space required for such a comprehensive discussion would be excessive, and consequently a limited number of cases only will be treated. These, however, will serve to illustrate the principles involved and the mode of attack, which then may be applied to the analysis of other cases, as need may arise. Symmetrical coordinates are not used in the simpler problems, which can be just as easily solved without resorting to this device. The method of symmetrical coordinates, however, is utilized in the more complicated cases, where it highly contributes to the ease and elegance with which solutions may be obtained.

Both the two-circuit and three-circuit banks treated in the following are assumed to consist of single-phase transformers. Also, the transformers in the various phases are assumed to be strictly identical as far as electrical characteristics are concerned.

Two-circuit Transformers. Y-Y Connection without Primary Neutral.—The Y-Y connection without neutral on the primary side is little used in practice, on account of its decidedly unsatisfactory operating features, even when the loading is only slightly unbalanced. This is due to the fact that the performance of this connection, when loaded unsymmetrically, to a considerable extent depends on the no-load characteristics of the transformers, particularly when the unbalance is pronounced. This connection therefore, is interesting from a theoretical standpoint since it represents the only case where it is *necessary* to take into account

the exciting current in order to arrive at anything like a solution at all.

1. *Single-phase Short-circuit, Line to Neutral.*—A single-phase short circuit, line to neutral, as indicated in Fig. 60, evidently represents the extreme case of unbalanced loading.

Since no secondary current is drawn from phases b and c, the primary current in these phases is *exciting current only.* The primary current in phase a, however, consists of an exciting

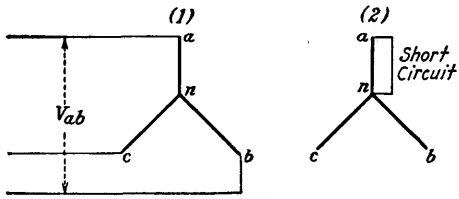

FIG. 60.—Single-phase line-to-neutral short circuit on the secondary side of a Y-Y-connected bank of transformers without primary neutral.

current plus a load component corresponding to the secondary short-circuit current of this phase.

The following equations may be written down immediately (see the general transformer equations in Chap. II):

$$V_{1a} = Z_1 I_{1a} - Z_2 I_{2a} \tag{1}$$
$$V_{1b} = Z_{11} I_{1b} + E_b \tag{2}$$
$$V_{1c} = Z_{11} I_{1c} + E_c \tag{3}$$
$$V_{2a} = 0 \tag{4}$$
$$V_{2b} = jx_{12} I_{1b} + E_b \tag{5}$$
$$V_{2c} = jx_{12} I_{1c} + E_c \tag{6}$$
$$E_a = - Z_{22} I_{2a} - jx_{12} I_{1a} \tag{7}$$
$$I_{1a} + I_{1b} + I_{1c} = 0 \tag{8}$$
$$V_{1ab} + V_{1bc} + V_{1ca} = 0 \tag{9}$$

These equations are perfectly general, holding for any arbitrary system of primary line voltages. Furthermore, they have been written down without imposing any simplifying assumption in regard to the magnetization characteristics of the transformers.

The primary line voltages may be expressed in terms of the phase voltages, as follows:

$$V_{1ab} = V_{1a} - V_{1b} = Z_1 I_{1a} - Z_{11} I_{1b} - Z_2 I_{2a} - E_b \tag{10}$$
$$V_{1bc} = V_{1b} - V_{1c} = Z_{11}(I_{1b} - I_{1c}) + E_b - E_c \tag{11}$$
$$V_{1ca} = V_{1c} - V_{1a} = Z_{11} I_{1c} - Z_1 I_{1a} + Z_2 I_{2a} + E_c \tag{12}$$

Subtracting equation (12) from equation (10) and making use of the current relation expressed by equation (8) gives

$$V_{1ab} - V_{1ca} = (2Z_1 + Z_{11})I_{1a} - 2Z_2I_{2a} - (E_b + E_c) \qquad (13)$$

In order to proceed further, it is necessary to introduce an empirical mathematical expression for the magnetization curve of the transformers; in other words, express the induced voltages as a function of the exciting currents. The well-known Fröhlich's equation[1]

$$\phi = \frac{ai}{b + i} \qquad (14)$$

relates the instantaneous flux and the instantaneous exciting current in a quite satisfactory manner for positive values of the argument. The instantaneous induced voltage is obtained by differentiation and is given by

$$e = (N \times 10^{-8})\frac{ab}{(b + i)^2}\frac{di}{dt} \qquad (15)$$

Since, however, it is not possible from equation (15) to establish a simple vector relation *between a voltage and current of the same frequency*, the use of Fröhlich's equation in this particular problem complicates the equations to such an extent that an explicit analytical solution for the currents in terms of the impressed voltages cannot be obtained. A cut-and-try process or else a simpler expression for the magnetization curve must be used. Assume a straight-line magnetization curve and a true quadrature relation between induced voltage and exciting current, the latter being equivalent to ignoring the core-loss current. The following equation embodies these assumptions:

$$E = jkI_e \qquad (16)$$

Then,

$$E_b + E_c = jk(I_{1b} + I_{1c}) = -jkI_{1a} \qquad (17)$$

which, inserted in equation (13), gives

$$V_{1ab} - V_{1ca} = (2Z_1 + Z_{11} + jk)I_{1a} - 2Z_2I_{2a} \qquad (18)$$

Splitting the current I_{1a} into its two components, the exciting current I_{1ae} and the load component $-I_{2a}$, equation (18) may be written

$$V_{1ab} - V_{1ca} = (2Z_1 + Z_{11} + jk)I_{1ae} - (2Z_{12} + Z_{11} + jk)I_{2a} \qquad (19)$$

[1] See, for instance, C. P. Steinmetz, "Theory and Calculation of Electric Circuits," p. 42, *et seq.*, McGraw-Hill Book Company, Inc., New York, 1917.

By expressing the induced voltage in transformer a (equation (7)) in terms of its exciting current, a relation between the exciting current and the load component in this phase is obtained.

$$\begin{aligned} jkI_{1ae} &= -Z_{22}I_{2a} - jx_{12}I_{1ae} + jx_{12}I_{2a} \\ &= -Z_2I_{2a} - jx_{12}I_{1ae} \end{aligned} \tag{20}$$

or

$$j(k + x_{12})I_{1ae} = -Z_2I_{2a} \tag{21}$$

Combining equations (19) and (21) gives

$$V_{1ab} - V_{1ca} = -\left[\frac{(2Z_1 + Z_{11} + jk)Z_2}{j(k + x_{12})} + 2Z_{12} + Z_{11} + jk\right] I_{2a} \tag{22}$$

From this equation, the current in the short circuit I_{2a} is obtained. If the primary line voltages are balanced, the voltage difference in equation (22) is equal to

$$V_{1ab} - V_{1ca} = \sqrt{3}V\diagdown 30^\circ \tag{23}$$

where V is the value of the line voltage. V_{1ab} has been taken as standard phase, the sequence being $a - b - c$.

Having determined the current I_{2a}, the exciting current I_{1ae} is found from equation (21). The total primary current of phase a is then calculated by

$$I_{1a} = I_{1ae} - I_{2a} \tag{24}$$

The currents in the other phases, as well as the primary voltages to neutral and the secondary voltages appearing across phases b and c, may now easily be computed by making use of the appropriate formulas.

The heaviest current will flow in phase a, at least when the line voltages are not unbalanced to an excessive degree, which very seldom will be the case in practice. The primary voltages to neutral will be decidedly unbalanced, the voltage across the short-circuited transformer (phase a) being very small compared to the others, due to the effect of the short circuit which causes the flux in this transformer to collapse to almost zero. In other words, the primary neutral point will travel from its symmetrical position toward line a, almost coinciding with the latter.

The accuracy of the results evidently depends on the correctness of the assumption in regard to the magnetization curve. If the transformers normally operate considerably below the knee, the straight-line representation may be proper and lead to correct results. Modern transformers, however, normally operate at a fairly high saturation, *i.e.*, at or slightly above the knee of the magnetization curve. In view of the large displacement of the

primary neutral point, causing considerably more than normal voltage to be impressed on the transformers in the non-short-circuited phases, large errors may result from the assumption of the magnetization curve's being a straight line.[1]

An approximate method applicable to all cases suggests itself, however. This method gives values of current which are conservatively high. If it is assumed that the neutral point is completely displaced, *i.e.*, lies on line a, then the line voltages V_{1ba} and V_{1ca} are impressed directly on the transformers in phases b and c, respectively.

$$V_{1b} = V_{1ba} = -V_{1ab} \text{ (approximately)} \tag{25}$$

$$V_{1c} = V_{1ca} \text{ (approximately)} \tag{26}$$

Assuming that the induced voltages do not differ appreciably from the impressed voltages, the exciting currents of transformers b and c (I_{1b} and I_{1c}) corresponding to the voltages given by equations (25) and (26) are read from a magnetization curve. The current in phase a is then given by

$$I_{1a} = -I_{2a} = -(I_{1b} + I_{1c}) \tag{27}$$

If the line voltages are balanced and $V_{1ab} = V\underline{/0}$ taken as standard phase, equations (25) and (26) may be written

$$V_{1b} = V\underline{/180^\circ} \text{ (approximately)} \tag{28}$$

$$V_{1c} = V\underline{/120^\circ} \text{ (approximately)} \tag{29}$$

These voltages are equal in magnitude and 60 deg. out of phase. Hence, the corresponding exciting currents will also be equal and

[1] In the solution above, the same straight-line magnetization curve, *i.e.*, the same value of k, has been used for all three transformers. Since the transformer in phase a operates at a very much lower flux density than the transformers in phases b and c, it would be appropriate and strictly more accurate to use a different and higher value of k for the former. This value (k_1) may be taken from the straight portion of the actual magnetization curve. The coefficient (k) for the two other transformers may be determined by means of a straight line drawn between the origin and a selected point well up on the actual magnetization curve. This point should correspond, as nearly as possible, to the voltages across phases b and c and assumes that these voltages are equal. This, however, will not be the case when the impressed line voltages are unbalanced. When the primary line is balanced, the voltages across the non-short-circuited transformers have strictly the same magnitude and are very nearly equal to the line voltage.

When different values of k are used, equations (21) and (22) become

$$j(k_1 + x_{12})I_{1ae} = -Z_2 I_{2a} \tag{21a}$$

and

$$V_{1ab} - V_{1ca} = -\left[\frac{2Z_1 + Z_{11} + jk)Z_2}{j(k_1 + x_{12})} + 2Z_{12} + Z_{11} + jk\right]I_{2a} \tag{22a}$$

phase-displaced 60 deg. The numerical value of the current in the short-circuited phase is, therefore,

$$I_{1a} = \sqrt{3}I_{1b} = \sqrt{3}I_{1c} \tag{30}$$

and is, as seen, very readily calculated.

2. *Single-phase Load, Line to Neutral.*—A single-phase load of impedance Z, connected line to neutral, as indicated in Fig. 61, will now be considered.

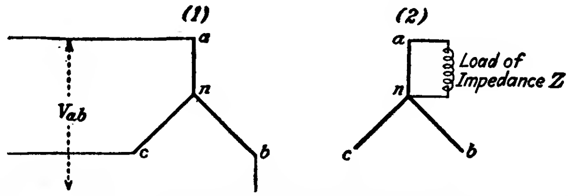

FIG. 61.—Single-phase line-to-neutral impedance load on a bank of Y-Y-connected transformers without primary neutral.

Equations (2), (3), (5), (6), (8), and (9), previously given, may be used also in this case. In addition may be written the general equations

$$V_{1a} = Z_{11}I_{1a} + jx_{12}I_{2a} + E_a \tag{31}$$

$$V_{2a} = Z_{22}I_{2a} + jx_{12}I_{1a} + E_a \tag{32}$$

which, in connection with

$$V_{2a} = -I_{2a}Z \tag{33}$$

give

$$V_{1a} = Z_1I_{1a} - (Z_2 + Z)I_{2a} \tag{34}$$

and

$$E_a = -(Z_{22} + Z)I_{2a} - jx_{12}I_{1a} \tag{35}$$

By a procedure similar to the one used in the previous case, an expression for the difference between the line voltages V_{1ab} and V_{1ca} may be formed. The result is

$$V_{1ab} - V_{1ca} = (2Z_1 + Z_{11})I_{1a} - 2(Z_2 + Z)I_{2a} - (E_b + E_c) \tag{36}$$

Again, in order to obtain an explicit analytical solution, straight-line magnetization curves will be assumed. Equation (36) may then be written

$$\begin{aligned} V_{1ab} - V_{1ca} &= (2Z_1 + Z_{11} + jk)I_{1a} - 2(Z_2 + Z)I_{2a} \\ &= (2Z_1 + Z_{11} + jk)I_{1ae} - \\ &\qquad (2Z_{12} + 2Z + Z_{11} + jk)I_{2a} \end{aligned} \tag{37}$$

Introducing the linear relation between induced voltage and exciting current, in equation (35), this reduces to

$$j(k + x_{12})I_{1ae} = -(Z_2 + Z)I_{2a} \tag{38}$$

which, combined with equation (37), gives

$$V_{1ab} - V_{1ca} = -\left[\frac{(2Z_1 + Z_{11} + jk)(Z_2 + Z)}{j(k + x_{12})} + 2Z_{12} + 2Z + Z_{11} + jk\right]I_{2a} \quad (39)$$

This equation may be solved for the load current I_{2a}. The exciting current of transformer a is then found from equation (38), and the two combined to give the primary current in phase a as given by equation (24). The other quantities of interest may readily be found, as discussed for the short-circuit case.

If the load impedance Z is large, so that the current drawn is small, and, as a consequence, the shift of the primary neutral point is not large, the assumption of a straight-line magnetization curve may not give rise to appreciable errors in the results. If, on the other hand, the load impedance is low, the solutions are subject to the same limitations as in the case of a dead short circuit, although to a somewhat lesser degree.[1] If greater accuracy is desired, cut-and-try methods must be resorted to.

Y-Δ Connection.—The Y-Δ connection with unbalanced load on the secondary can be handled without taking the exciting current into account with the same degree of precision as in any single-phase or symmetrical transformer problem where exciting currents are neglected. If desired, it is entirely practicable to introduce the effect of this current; for most engineering purposes, however, this refinement is unnecessary and will be neglected in the following analysis.

Considering quantities of fundamental frequency only, the position of the primary neutral point is fixed independent of the load of the secondary. This will be apparent from the following reasoning: The displacement of the neutral, if any, is equal to the uniphase voltage. If a uniphase voltage were present, a circulat-

[1] Different values of k may be used for the transformer in phase a and the transformers in phases b and c, also when a single-phase, line-to-neutral load is supplied. The procedure is the same as that discussed in the footnote on p. 105 for the line-to-neutral short circuit.

Since the voltages across phases b and c depend upon the load impedance, the straight line representing the magnetization characteristic of the transformers in these phases cannot be located as definitely as in the short-circuit case.

With different values of k, equations (38) and (39) become

$$j(k_1 + x_{12})I_{1ae} = -(Z_2 + Z)I_{2a} \quad (38a)$$

$$V_{1ab} - V_{1ac} = -\left[\frac{(2Z_1 + Z_{11} + jk)(Z_2 + Z)}{j(k_1 + x_{12})} + 2Z_{12} + 2Z + Z_{11} + jk\right]I_{2a} \quad (39a)$$

ing current would, of necessity, exist in the delta. Since the Y-connection, however, eliminates the possibility of any uniphase currents' flowing in the primary windings, no uniphase or circulatory current can flow in the secondary delta, because, in general, ignoring exciting currents, a current in a secondary winding, in order to exist, must have an equivalent component in the corresponding primary winding. Hence, since there are no uniphase currents, there can be no uniphase voltages and, consequently, no displacement of the neutral point.

It should be noted, however, that there is a triple frequency displacement of the neutral point, due to the third-harmonic component of the exciting current which circulates in the secondary delta. This third-harmonic voltage, however, is quite small, being equal to the secondary third-harmonic leakage-impedance drop.

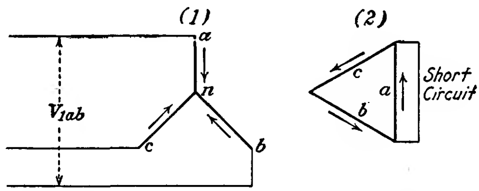

FIG. 62.—Single-phase line-to-line short circuit on the secondary side of a bank of Y-Δ-connected transformers without primary neutral.

1. *Single-phase Short Circuit.*—Consider a Y-Δ-connected bank, as shown in Fig. 62, with a dead single-phase short circuit on the secondary side.

Neglecting magnetizing current, the general equations for the primary and secondary terminal voltages may be written

$$V_{1a} = Z_1 I_{1a} + E_a \tag{40}$$
$$V_{1b} = Z_1 I_{1b} + E_b \tag{41}$$
$$V_{1c} = Z_1 I_{1c} + E_c \tag{42}$$
$$V_{2a} = -Z_2 I_{1a} + E_a \tag{43}$$
$$V_{2b} = -Z_2 I_{1b} + E_b \tag{44}$$
$$V_{2c} = -Z_2 I_{1c} + E_c \tag{45}$$

In addition to these may be written

$$I_{1a} + I_{1b} + I_{1c} = 0 \tag{46}$$
$$I_{2a} + I_{2b} + I_{2c} = 0 \tag{47}$$
$$V_{1ab} + V_{1bc} + V_{1ca} = 0 \tag{48}$$
$$V_{2a} + V_{2b} + V_{2c} = 0 \tag{49}$$

The equations as written above are not restricted to the case of a single-phase short circuit but hold for any type of load. In the case at hand, the secondary terminal voltage of the short-circuited phase equals zero, *i.e.*, $V_{2a} = 0$.

Hence,

$$E_a = Z_2 I_{1a} \tag{50}$$

and

$$V_{1a} = Z_{12} I_{1a} \tag{51}$$

The primary line voltages may now be expressed as follows:

$$V_{1ab} = V_{1a} - V_{1b} = Z_{12}I_{1a} - Z_1 I_{1b} - E_b \tag{52}$$
$$V_{1bc} = V_{1b} - V_{1c} = Z_1(I_{1b} - I_{1c}) + E_b - E_c \tag{53}$$
$$V_{1ca} = V_{1c} - V_{1a} = Z_1 I_{1c} - Z_{12}I_{1a} + E_c \tag{54}$$

Taking the difference between equations (52) and (54) gives

$$\begin{aligned} V_{1ab} - V_{1ca} &= 2Z_{12}I_{1a} - Z_1\,(I_{1b} + I_{1c}) - (E_b + E_c) \\ &= (2Z_{12} + Z_1)I_{1a} - (E_b + E_c) \end{aligned} \tag{55}$$

From equations (49), (44), and (45) is obtained

$$0 = V_{2b} + V_{2c} = -Z_2(I_{1b} + I_{1c}) + E_b + E_c = Z_2 I_{1a} + E_b + E_c \tag{56}$$

which gives

$$E_b + E_c = -Z_2 I_{1a} \tag{57}$$

Substituting this in equation (55) gives

$$V_{1ab} - V_{1ca} = (2Z_{12} + Z_1)I_{1a} + Z_2 I_{1a} = 3Z_{12}I_{1a} \tag{58}$$

The primary current of phase a is thus

$$I_{1a} = \frac{V_{1ab} - V_{1ca}}{3Z_{12}} \tag{59}$$

Evidently,

$$I_{2b} = I_{2c} \tag{60}$$

hence, also,

$$I_{1b} = I_{1c} \tag{61}$$

which, in connection with equation (46), gives

$$I_{1b} = I_{1c} = -\frac{I_{1a}}{2} \tag{62}$$

The total current in the secondary short circuit is

$$I = I_{2a} - I_{2b} = \frac{3I_{2a}}{2} = -\frac{V_{1ab} - V_{1ca}}{2Z_{12}} \tag{63}$$

If the primary line voltages are balanced of magnitude V, equations (59) and (63) may be written, using $V_{1ab} = V\underline{/0}$ as standard phase and assuming the phase sequence to be $a - b - c$,

$$I_{1a} = \frac{V\overline{\diagdown 30^\circ}}{\sqrt{3}Z_{12}} \tag{64}$$

$$I = -\frac{\sqrt{3}V\overline{\diagdown 30^\circ}}{2Z_{12}} \tag{65}$$

The equations show that the primary current carried by the short-circuited phase splits equally between the two others; in other words, the current in the two latter are of one-half the magnitude and 180 deg. out of phase with the current in the former. On the secondary side, the currents add up to a total short-circuit current of three-halves the value of current in the most heavily loaded phase. The calculations are, as seen, extremely simple. The only constant required is the equivalent (short-circuit) impedance of the transformers.

2. *Single-phase Load on Secondary.*—If the Y-Δ-connected bank is loaded with a single-phase load of impedance Z, as shown

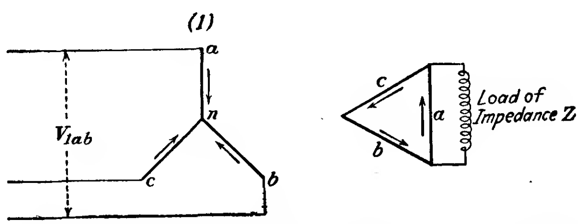

FIG. 63.—Single-phase impedance load on a bank of Y-Δ-connected transformers without primary neutral.

in Fig. 63, equations (40) to (49) inclusive may still be used. In addition, the following equations may be written:

$$V_{2a} = -(I_{2a} - I_{2b})Z = \tfrac{3}{2}I_{1a}Z \tag{66}$$

which, substituted in equation (43), gives

$$E_a = (Z_2 + \tfrac{3}{2}Z)I_{1a} \tag{67}$$

From this, in connection with equation (40), is obtained

$$V_{1a} \doteq (Z_{12} + \tfrac{3}{2}Z)I_{1a} \tag{68}$$

This equation may be compared with equation (51), which is the corresponding one when there is a single-phase short circuit on the secondary. Since the only difference is the addition of three-halves of the load impedance to the equivalent transformer impedance when the bank is loaded, it may be immediately inferred that, upon following the method of solution used in the previous case, final equations of the same form would be obtained, with the exception that $Z_{12} + \tfrac{3}{2}Z$ should be used instead of merely Z_{12}. Hence, making use of equations (59), (63), (64),

and (65), the solutions for the current in the most heavily loaded transformer, and for the load current, become

$$I_{1a} = \frac{V_{1ab} - V_{1ca}}{3(Z_{12} + \frac{3}{2}Z)} \tag{69}$$

$$I = -\frac{V_{1ab} - V_{1ca}}{2(Z_{12} + \frac{3}{2}Z)} \tag{70}$$

for the general case, and, when the primary line voltages are balanced,

$$I_{1a} = \frac{V\diagdown 30^\circ}{\sqrt{3}(Z_{12} + \frac{3}{2}Z)} \tag{71}$$

$$I = -\frac{\sqrt{3}V\diagdown 30^\circ}{2(Z_{12} + \frac{3}{2}Z)} \tag{72}$$

The relation between the currents in the various phases is evidently as in the short-circuited case and may be computed by equations previously given.

Three-circuit Transformers.—The general equations for the three-circuit transformer which were derived in Chap. II will be repeated here for reference.

$$V_{1a} = Z_{11}I_{1a} + jx_{12}I_{2a} + jx_{13}I_{3a} + E_a \tag{73}$$
$$V_{1b} = Z_{11}I_{1b} + jx_{12}I_{2b} + jx_{13}I_{3b} + E_b \tag{74}$$
$$V_{1c} = Z_{11}I_{1c} + jx_{12}I_{2c} + jx_{13}I_{3c} + E_c \tag{75}$$
$$V_{2a} = Z_{22}I_{2a} + jx_{12}I_{1a} + jx_{23}I_{3a} + E_a \tag{76}$$
$$V_{2b} = Z_{22}I_{2b} + jx_{12}I_{1b} + jx_{23}I_{3b} + E_b \tag{77}$$
$$V_{2c} = Z_{22}I_{2c} + jx_{12}I_{1c} + jx_{23}I_{3c} + E_c \tag{78}$$
$$V_{3a} = Z_{33}I_{3a} + jx_{13}I_{1a} + jx_{23}I_{2a} + E_a \tag{79}$$
$$V_{3b} = Z_{33}I_{3b} + jx_{13}I_{1b} + jx_{23}I_{2b} + E_b \tag{80}$$
$$V_{3c} = Z_{33}I_{3c} + jx_{13}I_{1c} + jx_{23}I_{2c} + E_c \tag{81}$$
$$V_{1a} - V_{2a} = Z_1I_{1a} - Z_2I_{2a} \tag{82}$$
$$V_{2a} - V_{3a} = Z_2I_{2a} - Z_3I_{3a} \tag{83}$$
$$V_{3a} - V_{1a} = Z_3I_{3a} - Z_1I_{1a} \tag{84}$$
$$V_{1b} - V_{2b} = Z_1I_{1b} - Z_2I_{2b} \tag{85}$$
$$V_{2b} - V_{3b} = Z_2I_{2b} - Z_3I_{3b} \tag{86}$$
$$V_{3b} - V_{1b} = Z_3I_{3b} - Z_1I_{1b} \tag{87}$$
$$V_{1c} - V_{2c} = Z_1I_{1c} - Z_2I_{2c} \tag{88}$$
$$V_{2c} - V_{3c} = Z_2I_{2c} - Z_3I_{3c} \tag{89}$$
$$V_{3c} - V_{1c} = Z_3I_{3c} - Z_1I_{1c} \tag{90}$$

Since the transformer bank itself represents a symmetrical circuit, these equations hold separately for the positive-, negative-, and zero-sequence (uniphase) components as well as for the actual voltages and currents.

Y-Δ-Y Connection.—It is a well-known fact that a bank of transformers connected in Y on the primary and secondary sides may be operated quite satisfactorily without primary neutral connection if supplied with a tertiary Δ-connected winding. The principal purpose of this winding, in this particular connection, is to provide a path of low impedance for the third-harmonic component of the exciting current, so that excessive distortion of the voltages is avoided. For certain types of unbalanced load and dissymmetrical short circuits, however, heavy circulating currents of fundamental frequency may be set up in the tertiary winding. These currents may often exceed by a wide margin the maximum current which can be carried without injury to the winding. Special relay protection for the tertiary is, therefore, usually

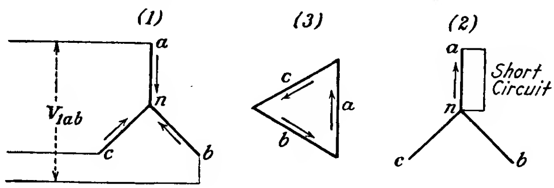

FIG. 64.—Single-phase line-to-neutral short circuit on the secondary side of a bank of Y-Δ-Y-connected transformers without primary neutral.

required, and it becomes of considerable importance to predetermine with precision the performance of such a bank when it supplies an unbalanced load or is subjected to dissymmetrical short circuits.

1. *Single-phase Short Circuit, Line to Neutral.*—Consider first a secondary single-phase short circuit, line to neutral, as shown in Fig. 64. To the equations which already have been given may be added the following holding for the particular connection being treated:

$$V_{1ab} + V_{1bc} + V_{1ca} = 0 \tag{91}$$

$$V_{3a} + V_{3b} + V_{3c} = 0 \tag{92}$$

Also, it should be remembered that, neglecting exciting currents, the sum of the current in the three windings of any one phase must be zero. Hence, without reference to phase,

$$I_1 + I_2 + I_3 = 0 \tag{93}$$

The uniphase voltage is, by definition, equal to one-third of the vector sum of the phase voltages and is, hence, for winding 1,

$$\begin{aligned} V_{10} &= \tfrac{1}{3}(V_{1a} + V_{1b} + V_{1c}) \\ &= Z_{11}I_{10} + jx_{12}I_{20} + jx_{13}I_{30} + E_0 \end{aligned} \tag{94}$$

Similarly, for the other windings,

$$V_{20} = Z_{22}I_{20} + jx_{12}I_{10} + jx_{23}I_{30} + E_0 \tag{95}$$
$$V_{30} = Z_{33}I_{30} + jx_{13}I_{10} + jx_{23}I_{20} + E_0 \tag{96}$$

V_{10} and V_{20} represent the displacement of the primary and secondary neutral points, respectively, from the no-load position corresponding to the impressed voltages. If the latter are balanced, the neutral point will, at no load, be located symmetrically in the equilateral triangle representing the line voltages.

Since winding 3 is a closed delta, $V_{30} = 0$. Introducing this in equation (96), an expression for E_0, the uniphase voltage induced per phase due to the flux exclusively existing in the iron core, is obtained.

$$E_0 = -Z_{33}I_{30} - jx_{13}I_{10} - jx_{23}I_{20} \tag{97}$$

Substituting equation (97) in equations (94) and (95) gives

$$V_{10} = (Z_{11} - jx_{13})I_{10} + j(x_{12} - x_{23})I_{20} - (Z_{33} - jx_{13})I_{30} \tag{98}$$
$$V_{20} = (Z_{22} - jx_{23})I_{20} + j(x_{12} - x_{13})I_{10} - (Z_{33} - jx_{23})I_{30} \tag{99}$$

The sum of the uniphase components in the three windings must be zero; hence,

$$I_{10} + I_{20} + I_{30} = 0 \tag{100}$$

Since there is no neutral connection on the primary side, however, no uniphase currents can flow in the primary windings. Therefore,

$$I_{20} = -I_{30} \tag{101}$$

This relation inserted in equations (98) and (99) gives

$$V_{10} = [Z_{33} + j(x_{12} - x_{23} - x_{13})]I_{20} = Z_3I_{20} \tag{102}$$
$$V_{20} = [Z_{33} - jx_{23} + Z_{22} - jx_{23}]I_{20} = Z_{23}I_{20} \tag{103}$$

From these equations, the displacement of the neutral points may readily be computed as soon as I_{20} has been determined.

The following equations may be written for the sum of the phase currents in the three windings:

$$I_{1a} + I_{1b} + I_{1c} = 0 \tag{104}$$
$$I_{2a} + I_{2b} + I_{2c} = 3I_{20} \tag{105}$$
$$I_{3a} + I_{3b} + I_{3c} = 3I_{30} \tag{106}$$

I_{2b} and I_{2c} are both zero; I_{2a} must, therefore, be equal to three times I_{20}. The sum of the positive- and negative-sequence currents in phase a of winding 2 will equal two times I_{20}. The sum of the positive-, negative-, and zero-sequence components in each phase of winding 2 is given by

$$I_{2a}^{+} + I_{2a}^{-} + I_{20} = 3I_{20} \tag{107}$$
$$I_{2b}^{+} + I_{2b}^{-} + I_{20} = 0 \tag{108}$$
$$I_{2c}^{+} + I_{2c}^{-} + I_{20} = 0 \tag{109}$$

These equations are evidently satisfied by the systems shown in Fig. 65. The primary positive- and negative-sequence phase voltages can readily be evaluated in terms of the line voltages either by analytical or graphical methods, previously described (Chap. III). If the primary line voltages are balanced, the magnitude of the positive-sequence voltages is equal to line voltage divided by $\sqrt{3}$, and the negative-sequence voltages are zero. The secondary voltages may be computed from the primary voltages by making use of equations (82), (85), and (88)

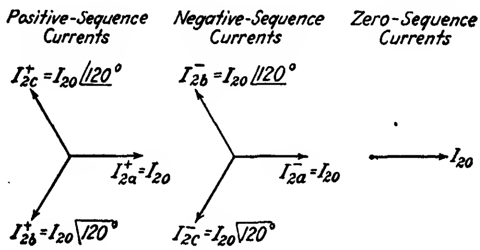

Fig. 65.—Vector diagram of positive-, negative-, and zero-sequence-current components in the secondary of the transformers in Fig. 64.

and applying them separately to each of the component systems. Thus, for phase a,

$$V_{2a}^{+} = V_{1a}^{+} - Z_1 I_{1a}^{+} + Z_2 I_{2a}^{+} \tag{110}$$
$$V_{2a}^{-} = V_{1a}^{-} - Z_1 I_{1a}^{-} + Z_2 I_{2a}^{-} \tag{111}$$
$$V_{20} = V_{10} - Z_1 I_{10} + Z_2 I_{20} \tag{112}$$

Since there can be no positive- and negative-sequence current in the tertiary winding, on account of the delta connection, the primary and secondary currents of these sequences are equal and opposite. Furthermore, I_{10} is zero. Equations (110), (111), and (112) then reduce to

$$V_{2a}^{+} = V_{1a}^{+} + Z_{12} I_{2a}^{+} \tag{113}$$
$$V_{2a}^{-} = V_{1a}^{-} + Z_{12} I_{2a}^{-} \tag{114}$$
$$V_{20} = V_{10} + Z_2 I_{20} \tag{115}$$

Since the secondary voltage of the short-circuited phase is zero, the following relation may be written:

$$V_{2a}^{+} + V_{2a}^{-} + V_{20} = 0 \tag{116}$$

which, in connection with equations (103), (107), (113), and (114), gives

$$V_{1a}^{+} + Z_{12} I_{20} + V_{1a}^{-} + Z_{12} I_{20} + Z_{23} I_{20} = 0 \tag{117}$$

Hence,

$$I_{20} = -\frac{V_{1a}^{+} + V_{1a}^{-}}{2Z_{12} + Z_{23}} \tag{118}$$

Also,

$$I_{2a} = 3I_{20} = -\frac{3(V_{1a}^{+} + V_{1a}^{-})}{2Z_{12} + Z_{23}} \tag{119}$$

$$I_{1a} = -2I_{20} = \frac{2(V_{1a}^{+} + V_{1a}^{-})}{2Z_{12} + Z_{23}} \tag{120}$$

$$I_{3a} = I_{30} = -I_{20} = \frac{V_{1a}^{+} + V_{1a}^{-}}{2Z_{12} + Z_{23}} \tag{121}$$

The other currents as well as the various voltages may now readily be determined by making use of the appropriate equations.

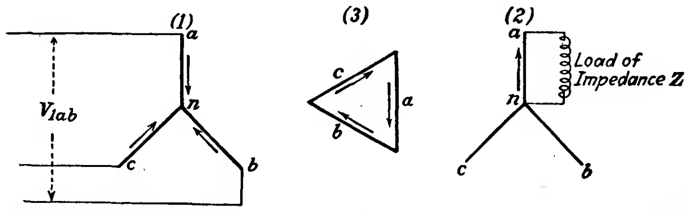

FIG. 66.—Single-phase line-to-neutral impedance load on a Y-Δ-Y-connected bank of transformers without primary neutral.

If the primary line voltages are balanced, of magnitude V and $V_{1ab} = V\underline{/0}$, the phase order being $a - b - c$, equations (118), (119), (120), and (121) reduce to

$$I_{20} = -\frac{V\diagdown 30^\circ}{\sqrt{3}(2Z_{12} + Z_{23})} \tag{122}$$

$$I_{2a} = -\frac{\sqrt{3}\,V\diagdown 30^\circ}{2Z_{12} + Z_{23}} \tag{123}$$

$$I_{1a} = \frac{2V\diagdown 30^\circ}{\sqrt{3}(2Z_{12} + Z_{23})} \tag{124}$$

$$I_{3a} = \frac{V\diagdown 30^\circ}{\sqrt{3}(2Z_{12} + Z_{23})} \tag{125}$$

2. *Single-phase Load, Line to Neutral.*—When the secondary is loaded with a single-phase load of impedance Z, connected line to neutral, as shown in Fig. 66, the analysis can be carried through exactly as for the single-phase line-to-neutral short circuit. Since, also, in this case, the secondary currents are zero in phases b and c, the system of currents illustrated in Fig. 65 is valid. The

general equations (73) to (90), as well as equations (91) to (115) inclusive, still hold.

Since the terminal voltage of phase a, winding 2, may be expressed in terms of the impedance drop in the load, equation (116) may be modified for the present case as follows:

$$V_{2a}^{+} + V_{2a}^{-} + V_{20} = -ZI_{2a} = -3ZI_{20} \qquad (126)$$

which, combined with equations (103), (107), (113), and (114), gives

$$V_{1a}^{+} + Z_{12}I_{20} + V_{1a}^{-} + Z_{12}I_{20} + Z_{23}I_{20} + 3ZI_{20} = 0 \qquad (127)$$

Hence,

$$I_{20} = -\frac{V_{1a}^{+} + V_{1a}^{-}}{2Z_{12} + Z_{23} + 3Z} \qquad (128)$$

The currents in the three windings of the loaded phase a are then given by

$$I_{2a} = 3I_{20} = -\frac{3(V_{1a}^{+} + V_{1a}^{-})}{2Z_{12} + Z_{23} + 3Z} \qquad (129)$$

$$I_{1a} = -2I_{20} = \frac{2(V_{1a}^{+} + V_{1a}^{-})}{2Z_{12} + Z_{23} + 3Z} \qquad (130)$$

$$I_{3a} = I_{30} = -I_{20} = \frac{V_{1a}^{+} + V_{1a}^{-}}{2Z_{12} + Z_{23} + 3Z} \qquad (131)$$

When the line voltages are balanced, these equations may, as in the previous case, be reduced, the result being

$$I_{20} = -\frac{V\backslash 30^\circ}{\sqrt{3}(2Z_{12} + Z_{23} + 3Z)} \qquad (132)$$

$$I_{2a} = -\frac{\sqrt{3}V\backslash 30^\circ}{2Z_{12} + Z_{23} + 3Z} \qquad (133)$$

$$I_{1a} = \frac{2V\backslash 30^\circ}{\sqrt{3}(2Z_{12} + Z_{23} + 3Z)} \qquad (134)$$

$$I_{3a} = \frac{V\backslash 30^\circ}{\sqrt{3}(2Z_{12} + Z_{23} + 3Z)} \qquad (135)$$

3. *Single-phase Short Circuit, Line to Line.*—Consider next a single-phase line-to-line short circuit on the secondary side, as indicated in Fig. 67. This problem actually reduces to a two-circuit transformer problem for the following reason: Since there is no neutral connection on either the primary or the secondary side, there can obviously be no uniphase currents in any of these circuits. Hence, no uniphase current can exist in the tertiary delta in which then, since it is unloaded, no current of fundamental frequency will flow. In this case, therefore, the two neutral points will remain fixed in spite of the short circuit.

The currents will contain positive- and negative-sequence components only. For the secondary currents may be written

$$I_{2a} = I_{2a}^{+} + I_{2a}^{-} = -I_{2b} \tag{136}$$
$$I_{2b} = I_{2b}^{+} + I_{2b}^{-} = -I_{2a} \tag{137}$$
$$I_{2c} = I_{2c}^{+} + I_{2c}^{-} = 0 \tag{138}$$

These equations are satisfied by the system shown in Fig. 68.

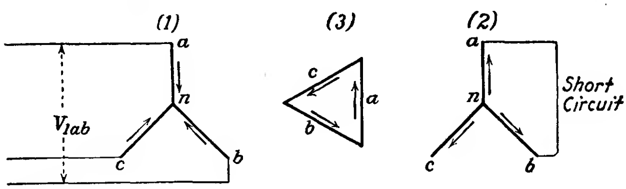

FIG. 67.—Single-phase line-to-line short circuit on the secondary side of a Y-Δ-Y-connected bank of transformers without primary neutral.

All components are equal and displaced 60 deg. from each other. Their magnitude is equal to the current in the short circuit divided by $\sqrt{3}$.

The solution of this case, however, is so simple and straightforward that nothing is actually gained by considering the component systems of currents and voltages. The solution,

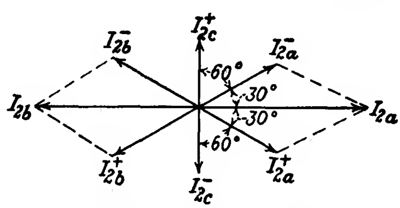

FIG. 68.—Vector diagram of positive- and negative-sequence current components in the secondary of the transformers in Fig. 67.

therefore, will be worked out below without resorting to this device.

Since the voltage across the short circuit is zero,

$$V_{2a} - V_{2b} = 0 \tag{139}$$

From equations (82) and (85) follow

$$V_{2a} = V_{1a} + Z_{12}I_{2a} \tag{140}$$
$$V_{2b} = V_{1b} + Z_{12}I_{2b} \tag{141}$$

which, inserted in equation (139), gives

$$V_{1a} - V_{1b} = -Z_{12}(I_{2a} - I_{2b}) \tag{142}$$

or,

$$V_{1ab} = -2Z_{12}I_{2a} \tag{143}$$

Hence,

$$I_{2a} = -I_{2b} = -\frac{V_{1ab}}{2Z_{12}} \tag{144}$$

$$I_{1a} = -I_{1b} = \frac{V_{1ab}}{2Z_{12}} \tag{145}$$

$$I_{1c} = -I_{2c} = 0 \tag{146}$$

It is seen that the solution, in this case, depends on the value of one of the impressed line voltages and is independent of whether the primary line voltage triangle is symmetrical or not. The only constant entering the solution is the equivalent impedance of windings 1 and 2 as determined by a simple short-circuit test.

4. *Single-phase Load, Line to Line.*—If a single-phase load is connected line to line on the secondary side, as shown in Fig. 69,

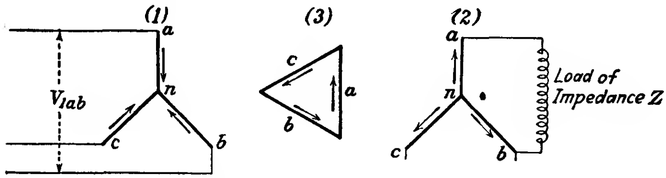

FIG. 69.—Single-phase line-to-line impedance load on a Y-Δ-Y-connected bank of transformers without primary neutral.

the solution is similar to the solution of the short-circuit case and is just as readily obtained. Equations (136), (137), (138), (140), and (141) hold unaltered, also in this case. Equation (139) may now be written

$$V_{2a} - V_{2b} = -I_{2a}Z \tag{147}$$

which, in connection with equations (140) and (141), gives

$$V_{1a} - V_{1b} = -Z_{12}(I_{2a} - I_{2b}) - ZI_{2a} \tag{148}$$

or,

$$V_{1ab} = -(2Z_{12} + Z)I_{2a} \tag{149}$$

Hence,

$$I_{2a} = -I_{2b} = -\frac{V_{1ab}}{2Z_{12} + Z} \tag{150}$$

$$I_{1a} = -I_{1b} = \frac{V_{1ab}}{2Z_{12} + Z} \tag{151}$$

$$I_{1c} = -I_{2c} = 0 \tag{152}$$

5. *Two Single-phase Short Circuits, Line to Neutral.*—The analysis of this case will be given as an interesting and instructive application of symmetrical coordinates, although, actually,

single-phase short circuits from line to neutral are not likely to occur in practice on two of the phases simultaneously. The circuit is shown in Fig. 70.

The secondary currents may be expressed by

$$I_{2a} = I_{2a}^{+} + I_{2a}^{-} + I_{20} \tag{153}$$

$$I_{2b} = I_{2a}^{+}\overline{\backslash 120^\circ} + I_{2a}^{-}\underline{/120^\circ} + I_{20} \tag{154}$$

$$I_{2c} = I_{2a}^{+}\underline{/120^\circ} + I_{2a}^{-}\overline{\backslash 120^\circ} + I_{20} = 0 \tag{155}$$

from which the following relation results:

$$I_{2a}^{+}\overline{\backslash 60^\circ} + I_{2a}^{-}\underline{/60^\circ} = I_{20} \tag{156}$$

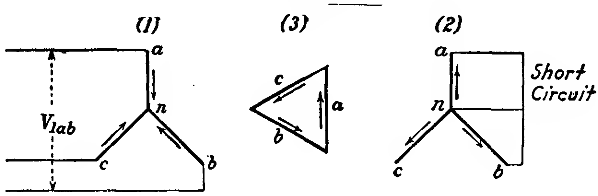

Fig. 70.—Two single-phase line-to-neutral short circuits on the secondary side of a Y-Δ-Y-connected bank of transformers without primary neutral.

Since the tertiary delta carries no positive- or negative-sequence currents, the secondary terminal voltages of these sequences of phases a and b may be written

$$V_{2a}^{+} = V_{1a}^{+} + Z_{12}I_{2a}^{+} \tag{157}$$

$$V_{2a}^{-} = V_{1a}^{-} + Z_{12}I_{2a}^{-} \tag{158}$$

$$V_{2b}^{+} = V_{1b}^{+} + Z_{12}I_{2a}^{+}\overline{\backslash 120^\circ} \tag{159}$$

$$V_{2b}^{-} = V_{1b}^{-} + Z_{12}I_{2a}^{-}\underline{/120^\circ} \tag{160}$$

The zero-sequence voltages on the secondary side are all equal to

$$V_{20} = Z_{23}I_{20} \tag{161}$$

as previously given by equation (103).

The secondary terminal voltages of the short-circuited phases are zero. Hence, from equations (157) to (161),

$$V_{2a}^{+} + V_{2a}^{-} + V_{20} = 0 = V_{1a}^{+} + V_{1a}^{-} + Z_{12}I_{2a}^{+} + Z_{12}I_{2a}^{-} + Z_{23}I_{20} \tag{162}$$

$$V_{2b}^{+} + V_{2b}^{-} + V_{20} = 0 = V_{1b}^{+} + V_{1b}^{-} + Z_{12}I_{2a}^{+}\overline{\backslash 120^\circ} + Z_{12}I_{2a}^{-}\underline{/120^\circ} + Z_{23}I_{20} \tag{163}$$

which, by addition, give

$$0 = V_{1a}^{+} + V_{1a}^{-} + V_{1b}^{+} + V_{1b}^{-} + Z_{12}(I_{2a}^{+}\overline{\backslash 60^\circ} + I_{2a}^{-}\underline{/60^\circ}) + 2Z_{23}I_{20} \tag{164}$$

Substituting equation (156) in equation (164), the zero-sequence current in the secondary is found in terms of the impressed voltages and the transformer constants.

$$I_{20} = -\frac{V_{1a}^{+} + V_{1a}^{-} + V_{1b}^{+} + V_{1b}^{-}}{Z_{12} + 2Z_{23}}$$
$$= -\frac{V_{1a}^{+}\backslash 60^\circ + V_{1a}^{-}/60^\circ}{Z_{12} + 2Z_{23}} \tag{165}$$

Simultaneous solution of equations (162) and (163) gives, for the positive- and negative-sequence currents,

$$I_{2a}^{+} = -\frac{V_{1a}^{+} + Z_{23}I_{20}/60^\circ}{Z_{12}} \tag{166}$$

$$I_{2a}^{-} = -\frac{V_{1a}^{-} + Z_{23}I_{20}\backslash 60^\circ}{Z_{12}} \tag{167}$$

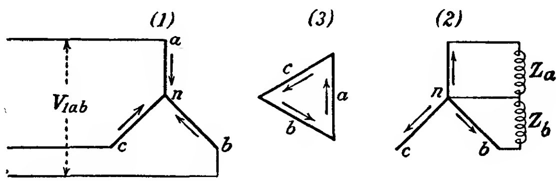

FIG. 71.—Two single-phase line-to-neutral impedance loads on a Y-Δ-Y-connected bank of transformers without primary neutral.

The short-circuit currents are now found by equations (153) and (154), and the current in the tertiary and phase a of the primary are given by

$$I_{30} = -I_{20} = \frac{V_{1a}^{+}\backslash 60^\circ + V_{1a}^{-}/60^\circ}{Z_{12} + 2Z_{23}} \tag{168}$$

$$I_{1a} = -(I_{2a}^{+} + I_{2a}^{-}) \tag{169}$$

If the impressed line voltages are balanced, the solutions for the secondary current components become

$$I_{20} = -\frac{V_{1ab}\backslash 90^\circ}{\sqrt{3}(Z_{12} + 2Z_{23})} \tag{170}$$

$$I_{2a}^{+} = -\frac{(Z_{12} + Z_{23})V_{1ab}\backslash 30^\circ}{\sqrt{3}Z_{12}(Z_{12} + 2Z_{23})} \tag{171}$$

$$I_{2a}^{-} = \frac{Z_{23}V_{1ab}\backslash 150^\circ}{\sqrt{3}Z_{12}(Z_{12} + 2Z_{23})} \tag{172}$$

6. *Two Single-phase Loads, Line to Neutral.*—When the transformer bank supplies two single-phase loads connected line to neutral, as shown in Fig. 71, equations (153) to (161) inclusive

are still valid. The secondary voltages of phases a and b may be equated to the impedance drops in the two loads. Thus,

$$V_{1a}^{+} + V_{1a}^{-} + Z_{12}I_{2a}^{+} + Z_{12}I_{2a}^{-} + Z_{23}I_{20} = - Z_aI_{2a}^{+} - Z_aI_{2a}^{-} - Z_aI_{20} \quad (173)$$

$$V_{1b}^{+} + V_{1b}^{-} + Z_{12}I_{2a}^{+}\overline{\backslash 120^\circ} + Z_{12}I_{2a}^{-}\underline{/120^\circ} + Z_{23}I_{20} = -Z_bI_{2a}^{+}\overline{\backslash 120^\circ} - Z_bI_{2a}^{-}\underline{/120^\circ} - Z_bI_{20} \quad (174)$$

Eliminating I_{20} by means of equation (156), these equations may be written

$$V_{1a}^{+} + V_{1a}^{-} = -[Z_{12} + Z_a + (Z_{23} + Z_a)\overline{\backslash 60^\circ}]I_{2a}^{+} - [Z_{12} + Z_a + (Z_{23} + Z_a)\underline{/60^\circ}]I_{2a}^{-} \quad (175)$$

$$V_{1b}^{+} + V_{1b}^{-} = -[(Z_{12} + Z_b)\overline{\backslash 120^\circ} + (Z_{23} + Z_b)\overline{\backslash 60^\circ}]I_{2a}^{+} - [(Z_{12} + Z_b)\underline{/120^\circ} + (Z_{23} + Z_b)\underline{/60^\circ}]I_{2a}^{-} \quad (176)$$

or,

$$V_{1a}^{+} + V_{1a}^{-} = -k_1I_{2a}^{+} - k_2I_{2a}^{-} \quad (177)$$

$$V_{1b}^{+} + V_{1b}^{-} = -k_3I_{2a}^{+} - k_4I_{2a}^{-} \quad (178)$$

Simultaneous solution of equations (177) and (178) gives

$$I_{2a}^{+} = \frac{k_4(V_{1a}^{+} + V_{1a}^{-}) - k_2(V_{1b}^{+} + V_{1b}^{-})}{k_2k_3 - k_1k_4} \quad (179)$$

$$I_{2a}^{-} = \frac{-k_3(V_{1a}^{+} + V_{1a}^{-}) + k_1(V_{1b}^{+} + V_{1b}^{-})}{k_2k_3 - k_1k_4} \quad (180)$$

which also may be written

$$I_{2a}^{+} = \frac{(k_4 + k_2\underline{/60^\circ})V_{1a}^{+} + (k_4 + k_2\overline{\backslash 60^\circ})V_{1a}^{-}}{k_2k_3 - k_1k_4} \quad (181)$$

$$I_{2a}^{-} = -\frac{(k_3 + k_1\underline{/60^\circ})V_{1a}^{+} + (k_3 + k_1\overline{\backslash 60^\circ})V_{1a}^{-}}{k_2k_3 - k_1k_4} \quad (182)$$

Having found the values of I_{2a}^{+} and I_{2a}^{-}, the zero-sequence current is found by equation (156). The other currents may be found as outlined for the short-circuited case. When the impressed line voltages are balanced, equations (181) and (182) reduce to

$$I_{2a}^{+} = \frac{(k_4 + k_2\underline{/60^\circ})V_{1ab}\overline{\backslash 30^\circ}}{\sqrt{3}(k_2k_3 - k_1k_4)} \quad (183)$$

$$I_{2a}^{-} = -\frac{(k_3 + k_1\underline{/60^\circ})V_{1ab}\overline{\backslash 30^\circ}}{\sqrt{3}(k_2k_3 - k_1k_4)} \quad (184)$$

7. *General Case of Unbalanced Three-phase Load, Line to Neutral.*—Let the secondary of the three-circuit transformer be loaded with three unequal impedances Z_a, Z_b, and Z_c, connected

line to neutral, as shown in Fig. 72. Although the solution of this case, as may be expected, is somewhat more complicated than any of the three-circuit transformer problems so far treated, it can be handled with comparative ease by the method of symmetrical coordinates. Without resorting to this scheme, the solution would be very laborious indeed.

It is convenient in this case of a general unbalanced load to make use of the equivalent positive-, negative-, and zero-sequence impedances of the load previously worked out (Chap. III):

$$Z_+ = \tfrac{1}{3}(Z_a + Z_b\underline{/120^\circ} + Z_c\overline{\backslash 120^\circ}) \tag{185}$$

$$Z_- = \tfrac{1}{3}(Z_a + Z_b\overline{\backslash 120^\circ} + Z_c\underline{/120^\circ}) \tag{186}$$

$$Z_0 = \tfrac{1}{3}(Z_a + Z_b + Z_c) \tag{187}$$

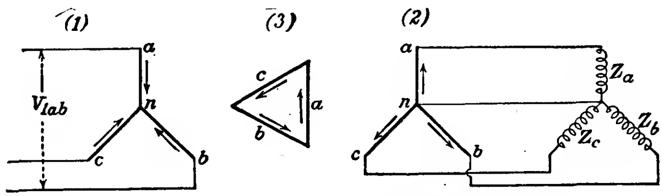

FIG. 72.—Unbalanced three-phase impedance load on a bank of Y-Δ-Y-connected transformers without primary neutral. The neutrals of the transformer secondaries and the load are interconnected.

Since the load impedances are supposed to be known, the value of these impedances can readily be evaluated. The general expressions for the secondary currents in terms of their positive-, negative-, and zero-sequence components have to be used in this problem, since there is no easy and obvious way by which the components can be immediately determined, as is usually the case in simpler problems. The secondary currents are, then,

$$I_{2a} = I_{2a}^+ + I_{2a}^- + I_{20} \tag{188}$$

$$I_{2b} = I_{2a}^+\overline{\backslash 120^\circ} + I_{2a}^-\underline{/120^\circ} + I_{20} \tag{189}$$

$$I_{2c} = I_{2a}^+\underline{/120^\circ} + I_{2a}^-\overline{\backslash 120^\circ} + I_{20} \tag{190}$$

Making use of equations (162), (166), and (167), in Chap. III, the components of the secondary terminal voltages of phase a are given by

$$V_{2a}^+ = -Z_0 I_{2a}^+ - Z_- I_{2a}^- - Z_+ I_{20} \tag{191}$$

$$V_{2a}^- = -Z_+ I_{2a}^+ - Z_0 I_{2a}^- - Z_- I_{20} \tag{192}$$

$$V_{20} = -Z_- I_{2a}^+ - Z_+ I_{2a}^- - Z_0 I_{20} \tag{193}$$

Similar equations may be written for the other phases but are not necessary for the solution. Substituting equation (191) in equation (113) and equation (192) in equation (114) gives

$$V_{1a}^{+} = -(Z_0 + Z_{12})I_{2a}^{+} - Z_{-}I_{2a}^{-} - Z_{+}I_{20} \qquad (194)$$
$$V_{1a}^{-} = -Z_{+}I_{2a}^{+} - (Z_0 + Z_{12})I_{2a}^{-} - Z_{-}I_{20} \qquad (195)$$

The displacements of the primary and secondary neutral points are also, in this case, given by equations (102) and (103). By combining equations (103) and (193), an expression for I_{20} in terms of the positive- and negative-sequence currents is obtained.

$$I_{20} = -\frac{Z_{-}I_{2a}^{+} + Z_{+}I_{2a}^{-}}{Z_0 + Z_{23}} \qquad (196)$$

Inserting equation (196) in equations (194) and (195) gives

$$V_{1a}^{+} = \left(\frac{Z_{+}Z_{-}}{Z_0 + Z_{23}} - Z_0 - Z_{12}\right)I_{2a}^{+} + \left(\frac{Z_{+}^2}{Z_0 + Z_{23}} - Z_{-}\right)I_{2a}^{-} \quad (197)$$
$$V_{1a}^{-} = \left(\frac{Z_{-}^2}{Z_0 + Z_{23}} - Z_{+}\right)I_{2a}^{+} + \left(\frac{Z_{+}Z_{-}}{Z_0 + Z_{23}} - Z_0 - Z_{12}\right)I_{2a}^{-} \quad (198)$$

These equations may now conveniently be written

$$V_{1a}^{+} = k_1 I_{2a}^{+} + k_2 I_{2a}^{-} \qquad (199)$$
$$V_{1a}^{-} = k_3 I_{2a}^{+} + k_1 I_{2a}^{-} \qquad (200)$$

where

$$k_1 = \frac{Z_{+}Z_{-}}{Z_0 + Z_{23}} - Z_0 - Z_{12} \qquad (201)$$
$$k_2 = \frac{Z_{+}^2}{Z_0 + Z_{23}} - Z_{-} \qquad (202)$$
$$k_3 = \frac{Z_{-}^2}{Z_0 + Z_{23}} - Z_{+} \qquad (203)$$

Eliminating I_{2a}^{+} and I_{2a}^{-}, respectively, between equations (199) and (200), the following explicit solutions for the secondary positive- and negative- sequence currents are obtained:

$$I_{2a}^{+} = \frac{k_1 V_{1a}^{+} - k_2 V_{1a}^{-}}{k_1^2 - k_2 k_3} \qquad (204)$$
$$I_{2a}^{-} = \frac{k_1 V_{1a}^{-} - k_3 V_{1a}^{+}}{k_1^2 - k_2 k_3} \qquad (205)$$

V_{1a}^{+} and V_{1a}^{-} are determined from the impressed line voltages. Hence, equations (204) and (205) can readily be solved. The uniphase current I_{20} is then computed by equation (196), and the actual secondary currents by equations (156), (157), and (158).

The primary currents are given by

$$I_{1a} = I_{1a}^{+} + I_{1a}^{-} = -(I_{2a}^{+} + I_{2a}^{-}) \tag{206}$$

$$I_{1b} = I_{1b}^{+} + I_{1b}^{-} = -(I_{2a}^{+}\overline{\backslash 120^\circ} + I_{2a}^{-}\underline{/120^\circ}) \tag{207}$$

$$I_{1c} = I_{1c}^{+} + I_{1c}^{-} = -(I_{2a}^{+}\underline{/120^\circ} + I_{2a}^{-}\overline{\backslash 120^\circ}) \tag{208}$$

and the tertiary current by

$$I_{3a} = I_{3b} = I_{3c} = I_{30} = -I_{20} \tag{209}$$

Having solved for the current, the various voltages, if desired, are easily determined by means of the appropriate formulas.

If the impressed line voltages are balanced, $V_{1a}^{+} = \frac{V}{\sqrt{3}}\overline{\backslash 30^\circ}$ while $V_{1a}^{-} = 0$. Equations (204) and (205) then reduce to

$$I_{2a}^{+} = \frac{k_1 \frac{V}{\sqrt{3}}\overline{\backslash 30^\circ}}{k_1^2 - k_2k_3} \tag{210}$$

$$I_{2a}^{-} = \frac{-k_3 \frac{V}{\sqrt{3}}\overline{\backslash 30^\circ}}{k_1^2 - k_2k_3} \tag{211}$$

The solutions just worked out for the general unbalanced load evidently also hold for special cases such as a single-phase load or a short circuit from line to neutral and also for a balanced three-phase load. It is inconvenient, however, to apply them to the former, since, when one or more of the load impedances are zero or infinite, they lead to indeterminate expressions which have to be especially evaluated. They are readily applicable to a balanced load, as illustrated below. This also gives a check on the correctness of the general formulas.

When the load is balanced,

$$Z_a = Z_b = Z_c = Z \tag{212}$$

Furthermore,

$$Z_+ = Z_- = 0 \text{ and } Z_0 = Z \tag{213}$$

Hence,

$$k_1 = -(Z + Z_{12}) \text{ and } k_2 = k_3 = 0 \tag{214}$$

Assuming balanced primary line voltages, equations (210), (211), and (196) give

$$I_{2a}^{+} = -\frac{V\overline{\backslash 30^\circ}}{\sqrt{3}(Z_{12} + Z)} \tag{215}$$

$$I_{2a}^{-} = 0 \qquad I_{20} = 0 \tag{216}$$

which evidently is correct and might have been written down immediately by inspection.

EXAMPLE 1

Statement of Problem

A three-phase, 115-kv., 60-cycle transmission line terminates at a substation in which the voltage is stepped down by a bank of transformers to approximately 38 kv. between lines in order to be transmitted over an underground system to a load center. The 20,000 kv.-a. transformer bank consists of single-phase three-circuit transformers connected Y-Δ-Y.

The nominal voltages of each single-phase transformer are as follows:

Winding 1	66,000 volts
Winding 2	22,000 volts
Winding 3	4,000 volts

The equivalent short-circuit reactances are

5.7 per cent between the 66-kv. and the 22-kv. windings
6.3 per cent between the 66-kv. and the 4-kv. windings
5.7 per cent between the 22-kv. and the 4-kv. windings

The 4,000-volt windings form the tertiary delta. The secondary neutral point is grounded.

Assuming the high-tension voltages to be strictly balanced and equal to 115 kv., calculate and compare the currents and the displacements of the neutral points for the following conditions of secondary short circuit:

1. Symmetrical short circuit.
2. Single-phase short circuit, line to neutral.
3. Single-phase short circuit, line to line.

Solution

Ratios of Transformation:

$$\frac{\text{Winding 1}}{\text{Winding 2}} = \frac{66}{22} = 3$$

$$\frac{\text{Winding 1}}{\text{Winding 3}} = \frac{66}{4} = 16.5$$

$$\frac{\text{Winding 2}}{\text{Winding 3}} = \frac{22}{4} = 5.5$$

The composite impedances (in this case, reactances only) are computed from the following formulas:

$$Z_1 = \frac{Z_{12} + Z_{13} - Z_{23}}{2} \qquad (a)$$

$$Z_2 = \frac{Z_{23} + Z_{12} - Z_{13}}{2} \qquad (b)$$

$$Z_3 = \frac{Z_{13} + Z_{23} - Z_{12}}{2} \qquad (c)$$

Hence,

$$x_1 = \frac{5.7 + 6.3 - 5.7}{2} = 3.15 \text{ per cent}$$

$$x_2 = \frac{5.7 + 5.7 - 6.3}{2} = 2.55 \text{ per cent}$$

$$x_3 = \frac{6.3 + 5.7 - 5.7}{2} = 3.15 \text{ per cent}$$

Balanced impressed voltage

$$V = 115{,}000 \text{ volts between lines}$$

Base current referred to high-tension side

$$I = \frac{20{,}000}{\sqrt{3} \times 115} = 100.4 \text{ amp.}$$

1. *Symmetrical Short Circuit.*

$$I_{1a} = \frac{V\diagdown 30^\circ}{\sqrt{3}Z_{12}} \tag{d}$$

Primary currents

$$|I_{1a}| = |I_{1b}| = |I_{1c}| = \frac{100.4 \times 100}{5.7} = 1{,}761 \text{ amp.}$$

Secondary currents

$$|I_{2a}| = |I_{2b}| = |I_{2c}| = 1{,}761 \times 3 = 5{,}283 \text{ amp.}$$

Tertiary currents

$$I_{3a} = I_{3b} = I_{3c} = 0$$

Displacements of neutral points

$$V_{10} = 0 \qquad V_{20} = 0$$

2. *Single-phase Short Circuit, Line to Neutral.*

$$I_{1a} = \frac{2V\diagdown 30^\circ}{\sqrt{3}(2Z_{12} + Z_{23})} \tag{e}$$

$$V_{10} = Z_3 I_{20} \tag{f}$$

$$V_{20} = Z_{23} I_{20} \tag{g}$$

Primary currents

$$|I_{1a}| = \frac{100.4 \times 100 \times 2}{2 \times 5.7 + 5.7} = 1{,}174 \text{ amp.}$$

$$|I_{1b}| = |I_{1c}| = \frac{|I_{1a}|}{2} = 587 \text{ amp.}$$

Secondary currents

$$|I_{2a}| = (1{,}174 + 587)3 = 5{,}283 \text{ amp.}$$

$$I_{2b} = I_{2c} = 0$$

Tertiary currents

$$|I_{3a}| = |I_{3b}| = |I_{3c}| = 587 \times 16.5 = 9{,}680 \text{ amp.}$$

Displacements of neutral points

$$V_{10} = \frac{3.15 \times 587}{100.4} = 18.4 \text{ per cent}$$

$$V_{20} = \frac{5.7 \times 587}{100.4} = 33.3 \text{ per cent}$$

Hence,

$$V_{10} = 12{,}210 \text{ volts} \qquad V_{20} = 7{,}380 \text{ volts}$$

3. *Single-phase Short Circuit, Line to Line.*

$$I_{1a} = \frac{V_{1ab}}{2Z_{12}} \tag{h}$$

Primary currents

$$|I_{1a}| = |I_{1b}| = \frac{\sqrt{3} \times 100.4 \times 100}{2 \times 5.7} = 1{,}525 \text{ amp.}$$

$$I_{1c} = 0$$

Secondary currents

$$|I_{2a}| = |I_{2b}| = 1{,}525 \times 3 = 4{,}575 \text{ amp.}$$
$$I_{2c} = 0$$

Tertiary currents

$$I_{3a} = I_{3b} = I_{3c} = 0$$

Displacements of neutral points

$$V_{10} = 0 \qquad V_{20} = 0$$

The results are gathered in Table VII so that they may be readily compared.

TABLE VII.—SHORT CIRCUIT ON A BANK OF Y-Δ-Y-CONNECTED TRANSFORMERS

(Example 1)

Quantities calculated	Symmetrical three-phase short circuit	Single-phase line-to-neutral short circuit on phase *a*	Single-phase line-to-line short circuit between phases *a* and *b*
Primary currents:			
Phase *a*	1,761 amp.	1,174 amp.	1,525 amp.
Phase *b*	1,761 amp.	587 amp.	1,525 amp.
Phase *c*	1,761 amp.	587 amp.	0 amp.
Secondary currents:			
Phase *a*	5,283 amp.	5,283 amp.	4,575 amp.
Phase *b*	5,283 amp.	0 amp.	4,575 amp.
Phase *c*	5,283 amp.	0 amp.	0 amp.
Tertiary currents:			
Phase *a*	0 amp.	9,680 amp.	0 amp.
Phase *b*	0 amp.	9,680 amp.	0 amp.
Phase *c*	0 amp.	9,680 amp.	0 amp.
Displacement of neutral points:			
Primary	0 volts	12,210 volts	0 volts
Secondary	0 volts	7,380 volts	0 volts

Δ-Δ-Y Connection.—When two of the windings of a three-circuit transformer are Δ-connected, two low-impedance paths are provided for the third-harmonic current. This connection, therefore, is very satisfactory from the standpoint of wave shape. When supplying unbalanced loads and dissymmetrical short circuits of such a nature that zero-sequence currents are produced, this connection is, in general, superior to the Y-Δ-Y connection in that it imposes a less severe duty on the tertiary delta.

1. *Single-phase Short-circuit, Line to Neutral.*—Consider a secondary single-phase short circuit, line to neutral, as shown in Fig. 73.

Obviously, no zero-sequence voltage can appear across the terminals of the Δ-connected windings, and the following equations may be written:

$$V_{10} - V_{20} = Z_1 I_{10} - Z_2 I_{20} = -V_{20} \tag{217}$$
$$V_{20} - V_{30} = Z_2 I_{20} - Z_3 I_{30} = V_{20} \tag{218}$$
$$V_{30} - V_{10} = Z_3 I_{30} - Z_1 I_{10} = 0 \tag{219}$$
$$I_{10} + I_{20} + I_{30} = 0 \tag{220}$$

Eliminating I_{30} by substituting equation (220) in equation (219) gives

$$I_{10} = -\frac{Z_3}{Z_1 + Z_3} I_{20} = -\frac{Z_3}{Z_{13}} I_{20} \tag{221}$$

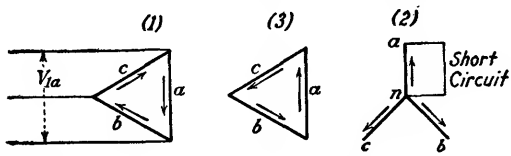

FIG. 73.—Single-phase line-to-neutral short circuit on the secondary side of a Δ-Δ-Y-connected bank of transformers.

which, inserted in equation (217), gives

$$V_{20} = \left(Z_2 + \frac{Z_1 Z_3}{Z_{13}}\right) I_{20} \tag{222}$$

The relative value of current components in the secondary (winding 2) are the same as for the Y-Δ-Y connection with a single-phase line-to-neutral short circuit. These components are illustrated in Fig. 65. Hence, also, in this case, the secondary current in the short-circuited phase is equal to three times the zero-sequence component. Positive- and negative-sequence currents exist only in the primary and secondary windings. Hence, for phase a,

$$V_{2a}^{+} = V_{1a}^{+} - Z_1 I_{1a}^{+} + Z_2 I_{2a}^{+} = V_{1a}^{+} + Z_{12} I_{2a}^{+} \tag{223}$$
$$V_{2a}^{-} = V_{1a}^{-} - Z_1 I_{1a}^{-} + Z_2 I_{2a}^{-} = V_{1a}^{-} + Z_{12} I_{2a}^{-} \tag{224}$$

The total secondary voltage of the short-circuited phase being zero, addition of equations (222), (223), and (224) gives

$$0 = V_{1a}^{+} + V_{1a}^{-} + \left(2Z_{12} + Z_2 + \frac{Z_1 Z_3}{Z_{13}}\right) I_{20} \tag{225}$$

from which

$$I_{20} = -\frac{V_{1a}^{+} + V_{1a}^{-}}{2Z_{12} + Z_2 + \dfrac{Z_1 Z_3}{Z_{13}}} = -\frac{V_{1a}}{2Z_{12} + Z_2 + \dfrac{Z_1 Z_3}{Z_{13}}} \tag{226}$$

The short-circuit current becomes

$$I_{2a} = 3I_{20} = -\frac{3V_{1a}}{2Z_{12} + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} \tag{227}$$

Using equation (221), the zero-sequence current in the primary is obtained as

$$I_{10} = \frac{Z_3V_{1a}}{Z_{13}\left(2Z_{12} + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)} \tag{228}$$

Also,

$$I_{3a} = I_{30} = -(I_{10} + I_{20}) = -\frac{Z_1}{Z_{13}}I_{20}$$

$$= \frac{Z_1V_{1a}}{Z_{13}\left(2Z_{12} + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)} \tag{229}$$

$$I_{1a} = -2I_{20} + I_{10} = \frac{(2Z_{13} + Z_3)V_{1a}}{Z_{13}\left(2Z_{12} + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)} \tag{230}$$

When the primary line voltages are balanced, the negative-sequence voltage (V_{1a}^-) is zero and the positive-sequence voltage

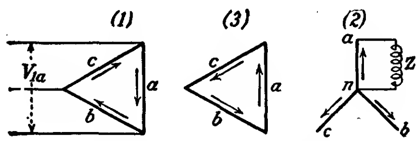

FIG. 74.—Single-phase line-to-neutral impedance load on a Δ-Δ-Y-connected bank of transformers.

(V_{1a}^+) becomes directly equal to the balanced line voltage (V). Since the currents, however, depend only on the line voltage impressed on phase a and are independent of the values of the other line voltages, the solutions remain the same.

2. *Single-phase Load, Line to Neutral.*—Consider the secondary loaded by a single-phase load of impedance Z, connected line to neutral, as shown in Fig. 74. Equations (217) to (224) inclusive hold also in this case. Equating the secondary terminal voltage of phase a to the drop across the load impedance gives

$$V_{2a}^+ + V_{2a}^- + V_{20} = V_{1a}^+ + V_{1a}^- + \left(2Z_{12} + Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)I_{20} = -3ZI_{20} \tag{231}$$

Hence,

$$I_{20} = -\frac{V_{1a}^{+} + V_{1a}^{-}}{2Z_{12} + Z_2 + \frac{Z_1 Z_3}{Z_{13}} + 3Z} = -\frac{V_{1a}}{2Z_{12} + Z_2 + \frac{Z_1 Z_3}{Z_{13}} + 3Z} \quad (232)$$

The load current is, consequently, given by

$$I_{2a} = 3I_{20} = -\frac{3V_{1a}}{2Z_{12} + Z_2 + \frac{Z_1 Z_3}{Z_{13}} + 3Z} \quad (233)$$

The primary zero-sequence current, the tertiary current, and the total current in phase a of the primary are given by

$$I_{10} = -\frac{Z_3}{Z_{13}} I_{20} = \frac{Z_3 V_{1a}}{Z_{13}\left(2Z_{12} + Z_2 + \frac{Z_1 Z_3}{Z_{13}} + 3Z\right)} \quad (234)$$

$$I_{3a} = I_{30} = -\frac{Z_1}{Z_{13}} I_{20} = \frac{Z_1 V_{1a}}{Z_{13}\left(2Z_{12} + Z_2 + \frac{Z_1 Z_3}{Z_{13}} + 3Z\right)} \quad (235)$$

$$I_{1a} = -2I_{20} + I_{10} = \frac{(2Z_{13} + Z_3)V_{1a}}{Z_{13}\left(2Z_{12} + Z_2 + \frac{Z_1 Z_3}{Z_{13}} + 3Z\right)} \quad (236)$$

3. *Single-phase Short Circuit, Line to Line.*—When a single-phase line-to-line short circuit occurs on the secondary side, as

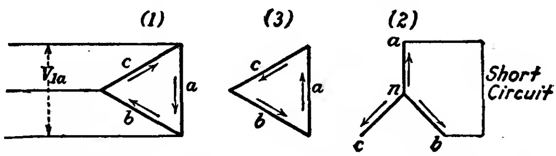

FIG. 75.—Single-phase line-to-line short circuit on the secondary side of a Δ-Δ-Y-connected bank of transformers.

shown in Fig. 75, the tertiary delta can carry no current of fundamental frequency. This is obviously true, since neither the primary nor the secondary currents can contain zero-sequence components. Very little is gained in this case by resolving the voltages and currents into symmetrical components, and the solution will be worked out without resorting to this device.

The secondary voltages of phases a and b may be written

$$V_{2a} = V_{1a} + Z_{12} I_{2a} \quad (237)$$

$$V_{2b} = V_{1b} + Z_{12} I_{2b} \quad (238)$$

Equating the voltage across the short circuit to zero gives

$$V_{2a} - V_{2b} = 0 = V_{1a} - V_{1b} + 2I_{2a}Z_{12} \tag{239}$$

from which

$$I_{2a} = -I_{2b} = -\frac{V_{1a} - V_{1b}}{2Z_{12}} \tag{240}$$

Also,

$$I_{1a} = -I_{1b} = \frac{V_{1a} - V_{1b}}{2Z_{12}} \tag{241}$$

$$I_{1c} = -I_{2c} = 0 \tag{242}$$

When the primary line voltages are balanced, the voltage difference in the above equations may be written, using the voltage of phase a as reference vector,

$$V_{1a} - V_{1b} = \sqrt{3}V_{1a}\underline{/30^\circ} \tag{243}$$

In this case, the currents are given by

$$I_{2a} = -I_{2b} = -\frac{\sqrt{3}V_{1a}\underline{/30^\circ}}{2Z_{12}} \tag{244}$$

$$I_{1a} = -I_{1b} = \frac{\sqrt{3}V_{1a}\underline{/30^\circ}}{2Z_{12}} \tag{245}$$

4. *Single-phase Load, Line to Line.*—If a single-phase load is connected line to line, as indicated in Fig. 76, the solution can be

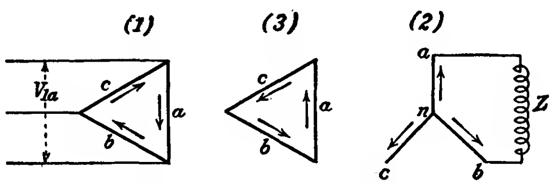

FIG. 76.—Single-phase line-to-line impedance load on a Δ-Δ-Y-connected bank of transformers.

worked out exactly as in the preceding case. Equations (237) and (238) still hold. Equation (239) may now be written

$$V_{2a} - V_{2b} = -ZI_{2a} = V_{1a} - V_{1b} + 2Z_{12}I_{2a} \tag{246}$$

Hence,

$$I_{2a} = -I_{2b} = -\frac{V_{1a} - V_{1b}}{2Z_{12} + Z} \tag{247}$$

$$I_{1a} = -I_{1b} = \frac{V_{1a} - V_{1b}}{2Z_{12} + Z} \tag{248}$$

$$I_{1c} = -I_{2c} = 0 \tag{249}$$

When the impressed line voltages are balanced, the currents become

$$I_{2a} = -I_{2b} = -\frac{\sqrt{3}V_{1a}\underline{/30°}}{2Z_{12} + Z} \qquad (250)$$

$$I_{1a} = -I_{1b} = \frac{\sqrt{3}V_{1a}\underline{/30°}}{2Z_{12} + Z} \qquad (251)$$

5. *Two Single-phase Short Circuits, Line to Neutral.*—Consider next single-phase short circuits from line to neutral on two phases, as shown in Fig. 77.

Equations (153) to (160) inclusive are valid in this case. Equation (222) gives the relation between the zero-sequence

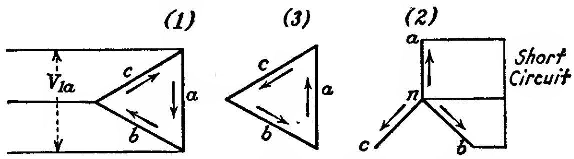

FIG. 77.—Two single-phase line-to-neutral short circuits on the secondary side of a Δ-Δ-Y-connected bank of transformers.

voltage and current in the secondary windings. Equating the secondary voltages of the short-circuited phases to zero gives

$$0 = V_{1a}^{+} + V_{1a}^{-} + Z_{12}I_{2a}^{+} + Z_{12}I_{2a}^{-} + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)I_{20} \qquad (252)$$

$$0 = V_{1b}^{+} + V_{1b}^{-} + Z_{12}I_{2a}^{+}\overline{\backslash 120°} + Z_{12}I_{2a}^{-}\underline{/120°} + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)I_{20} \qquad (253)$$

Adding equation (252) and equation (253) gives

$$0 = V_{1a}^{+} + V_{1a}^{-} + V_{1b}^{+} + V_{1b}^{-} + Z_{12}I_{2a}^{+}\overline{\backslash 60°} + Z_{12}I_{2a}^{-}\underline{/60°} + 2\left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)I_{20} \qquad (254)$$

Eliminating the positive- and negative-sequence current (using equation (156)), this equation reduces to

$$0 = V_{1a}^{+} + V_{1a}^{-} + V_{1b}^{+} + V_{1b}^{-} + \left[Z_{12} + 2\left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)\right]I_{20} \qquad (255)$$

from which

$$I_{20} = -\frac{V_{1a}^{+}\overline{\backslash 60°} + V_{1a}^{-}\underline{/60°}}{Z_{12} + 2\left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)} = -\frac{V_{1a} + V_{1b}}{Z_{12} + 2\left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)} \qquad (256)$$

By simultaneous solution of equations (252) and (253), the positive- and negative-sequence currents are obtained, as

$$I_{2a}^{+} = -\frac{V_{1a}^{+} + \left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)I_{20}\underline{/60^\circ}}{Z_{12}} =$$

$$-\frac{\dfrac{1}{\sqrt{3}}(V_{1a} + V_{1b}\underline{/60^\circ})\underline{/30^\circ} + \left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)I_{20}\underline{/60^\circ}}{Z_{12}} \tag{257}$$

$$I_{2a}^{-} = -\frac{V_{1a}^{-} + \left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)I_{20}\overline{\backslash 60^\circ}}{Z_{12}} =$$

$$-\frac{\dfrac{1}{\sqrt{3}}(V_{1a} + V_{1b}\overline{\backslash 60^\circ})\overline{\backslash 30^\circ} + \left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)I_{20}\overline{\backslash 60^\circ}}{Z_{12}} \tag{258}$$

The zero-sequence currents in the primary and tertiary windings are given by

$$I_{10} = -\frac{Z_3}{Z_{13}}I_{20} = \frac{Z_3(V_{1a}^{+}\overline{\backslash 60^\circ} + V_{1a}^{-}\underline{/60^\circ})}{Z_{13}\left[Z_{12} + 2\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)\right]} =$$

$$\frac{Z_3(V_{1a} + V_{1b})}{Z_{13}\left[Z_{12} + 2\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)\right]} \tag{259}$$

$$I_{30} = I_{3a} = -\frac{Z_1}{Z_{13}}I_{20} = \frac{Z_1(V_{1a}^{+}\overline{\backslash 60^\circ} + V_{1a}^{-}\underline{/60^\circ})}{Z_{13}\left[Z_{12} + 2\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)\right]} =$$

$$\frac{Z_1(V_{1a} + V_{1b})}{Z_{13}\left[Z_{12} + 2\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)\right]} \tag{260}$$

The secondary short-circuit currents are found by equations (153) and (154). The primary currents in the three phases are given by

$$I_{1a} = -(I_{2a}^{+} + I_{2a}^{-}) + I_{10} \tag{261}$$

$$I_{1b} = -(I_{2a}^{+}\overline{\backslash 120^\circ} + I_{2a}^{-}\underline{/120^\circ}) + I_{10} \tag{262}$$

$$I_{1c} = -(I_{2a}^{+}\underline{/120^\circ} + I_{2a}^{-}\overline{\backslash 120^\circ}) + I_{10} \tag{263}$$

When the primary line voltages are balanced, the equations for the secondary current components reduce to

$$I_{20} = -\frac{V_{1a}\overline{\backslash 60^\circ}}{Z_{12} + 2\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)} \tag{264}$$

$$I_{2a}^{+} = -\frac{\left(Z_{12} + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)V_{1a}}{Z_{12}\left[Z_{12} + 2\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)\right]} \tag{265}$$

$$I_{2a}^{-} = \frac{\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)V_{1a}\overline{\backslash 120^\circ}}{Z_{12}\left[Z_{12} + 2\left(Z_2 + \dfrac{Z_1Z_3}{Z_{13}}\right)\right]} \tag{266}$$

6. *Two Single-phase Loads, Line to Neutral.*—Let the transformer bank supply two single-phase loads of impedances Z_a and Z_b, respectively, connected line to neutral, as shown in Fig. 78.

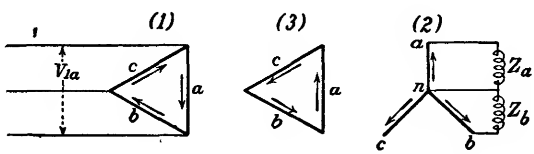

FIG. 78.—Two single-phase line-to-neutral impedance loads on a Δ-Δ-Y-connected bank of transformers.

The same equations which were referred to in the treatment above of the two single-phase line-to-neutral short circuits are applicable. The secondary terminal voltages of the loaded phases may now be equated to the drops in the load impedances, giving

$$V_{1a}^{+} + V_{1a}^{-} + Z_{12}I_{2a}^{+} + Z_{12}I_{2a}^{-} + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)I_{20} = -Z_a(I_{2a}^{+} + I_{2a}^{-} + I_{20}) \tag{267}$$

$$V_{1b}^{+} + V_{1b}^{-} + Z_{12}I_{2a}^{+}\overline{\backslash 120^\circ} + Z_{12}I_{2a}^{-}\underline{/120^\circ} + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}}\right)I_{20} = -Z_b(I_{2a}^{+}\overline{\backslash 120^\circ} + I_{2a}^{-}\underline{/120^\circ} + I_{20}) \tag{268}$$

Substituting for I_{20} in terms of I_{2a}^{+} and I_{2a}^{-}, by means of equation (156), these equations change to

$$V_{1a}^{+} + V_{1a}^{-} = -\left[Z_{12} + Z_a + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}} + Z_a\right)\diagdown 60^\circ\right] I_{2a}^{+} -$$

$$\left[Z_{12} + Z_a + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}} + Z_a\right)\underline{/60^\circ}\right] I_{2a}^{-} \quad (269)$$

$$V_{1b}^{+} + V_{1b}^{-} = -\left[(Z_{12} + Z_b)\diagdown 120^\circ + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}} + Z_b\right)\diagdown 60^\circ\right] I_{2a}^{+}$$

$$-\left[(Z_{12} + Z_b)\underline{/120^\circ} + \left(Z_2 + \frac{Z_1Z_3}{Z_{13}} + Z_b\right)\underline{/60^\circ}\right] I_{2a}^{-} \quad (270)$$

which may be written

$$V_{1a} = V_{1a}^{+} + V_{1a}^{-} = -k_1I_{2a}^{+} - k_2I_{2a}^{-} \quad (271)$$

$$V_{1b} = V_{1b}^{+} + V_{1b}^{-} = -k_3I_{2a}^{+} - k_4I_{2a}^{-} \quad (272)$$

Simultaneous solution of these equations gives the secondary positive- and negative-sequence currents. Since they are identical in form to equations (177) and (178), the solutions are given by equations (179) and (180) or by equations (181) and (182). When the positive- and negative-sequence currents are determined, the zero-sequence current in the secondary is found by equation (156). The currents in the other windings are determined as outlined for the preceding case.

When the primary line voltages are balanced, the secondary positive- and negative-sequence currents become

$$I_{2a}^{+} = \frac{(k_4 + k_2\underline{/60^\circ})V_{1a}}{k_2k_3 - k_1k_4} \quad (273)$$

$$I_{2a}^{-} = \frac{(k_3 + k_1\underline{/60^\circ})V_{1a}}{k_2k_3 - k_1k_4} \quad (274)$$

7. *General Case of Unbalanced Three-phase Load, Line to Neutral.*—Consider, finally, the transformer bank loaded with three unequal impedances Z_a, Z_b, and Z_c, connected line to neutral, as the circuit diagram (Fig. 79) indicates.

Making use of the equivalent positive-, negative-, and zero-sequence impedance of the unbalanced load, this problem can be handled by the same general method used in connection with the unbalanced three-phase load on the Y-Δ-Y-connected transformer bank. Evidently, equations (185) to (195) inclusive apply also in this case.

The zero-sequence voltage of the secondary is given by equation (222). Equating this to equation (193), the secondary zero-

sequence current is obtained in terms of the positive- and negative-sequence currents, as follows:

$$I_{20} = -\frac{Z_-I_{2a}^+ + Z_+I_{2a}^-}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} \tag{275}$$

Inserting this in equations (194) and (195) gives

$$V_{1a}^+ = \left[\frac{Z_+Z_-}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} - Z_0 - Z_{12}\right]I_{2a}^+ + \left[\frac{Z_+^2}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} - Z_-\right]I_{2a}^- \tag{276}$$

$$V_{1a}^- = \left[\frac{Z_-^2}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} - Z_+\right]I_{2a}^+ + \left[\frac{Z_+Z_-}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} - Z_0 - Z_{12}\right]I_{2a}^- \tag{277}$$

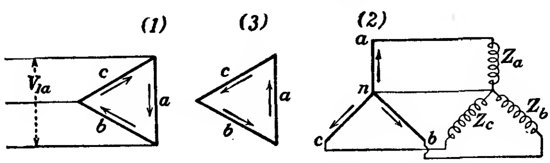

FIG. 79.—Unbalanced three-phase impedance load on a bank of Δ-Δ-Y-connected transformers. The neutrals of the transformer secondaries and the load are interconnected.

which may be written

$$V_{1a}^+ = k_1I_{2a}^+ + k_2I_{2a}^- \tag{278}$$
$$V_{1a}^- = k_3I_{2a}^+ + k_1I_{2a}^- \tag{279}$$

where

$$k_1 = \frac{Z_+Z_-}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} - Z_0 - Z_{12} \tag{280}$$

$$k_2 = \frac{Z_+^2}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} - Z_- \tag{281}$$

$$k_3 = \frac{Z_-^2}{Z_0 + Z_2 + \dfrac{Z_1Z_3}{Z_{13}}} - Z_+ \tag{282}$$

Equations (278) and (279) are similar to equations (199) and (200). The secondary positive- and negative-sequence currents, therefore, are given by equations (204) and (205). The secondary zero-sequence currents are determined by equation (275).

The primary zero-sequence current is given by equation (221) and is, hence,

$$I_{10} = -\frac{Z_3}{Z_{13}} I_{20} = \frac{Z_3(Z_- I_{2a}^+ + Z_+ I_{2a}^-)}{Z_{13}\left(Z_0 + Z_2 + \frac{Z_1 Z_3}{Z_{13}}\right)} \tag{283}$$

The total primary currents are

$$I_{1a} = -(I_{2a}^+ + I_{2a}^-) + I_{10} \tag{284}$$

$$I_{1b} = -(I_{2a}^+ \diagdown 120^\circ + I_{2a}^- \underline{/120^\circ}) + I_{10} \tag{285}$$

$$I_{1c} = -(I_{2a}^+ \underline{/120^\circ} + I_{2a}^- \diagdown 120^\circ) + I_{10} \tag{286}$$

and the tertiary current becomes

$$I_{3a} = I_{3b} = I_{3c} = I_{30} = -\frac{Z_1}{Z_{13}} I_{20} = \frac{Z_1(Z_- I_{2a}^+ + Z_+ I_{2a}^-)}{Z_{13}\left(Z_0 + Z_2 + \frac{Z_1 Z_3}{Z_{13}}\right)} \tag{287}$$

If the impressed line voltages are balanced, $V_{1a}^+ = V$ and $V_{1a}^- = 0$. The solutions for the secondary positive- and negative-sequence currents then reduce to

$$I_{2a}^+ = \frac{k_1 V}{k_1^2 - k_2 k_3} \tag{288}$$

$$I_{2a}^- = \frac{-k_3 V}{k_1^2 - k_2 k_3} \tag{289}$$

CHAPTER V

INDUCTION MOTORS ON UNBALANCED VOLTAGES SINGLE-PHASE OPERATION

General Discussion.—When unbalanced voltages are impressed on a three-phase induction motor, the currents which it takes will also be unbalanced. Assuming that the motor is either Δ-connected or Y-connected without neutral, which is always the case in practice, no uniphase currents of impressed frequency can possibly flow. The unbalanced current vectors, therefore, form a closed triangle and may be decomposed into two balanced systems of positive and negative phase sequence, respectively.

It is a well-known fact that, when the currents in a polyphase induction motor are balanced, a revolving field is set up in the air gap. This field, which, in the ideal case, has a sinusoidal space distribution, revolves with synchronous speed with respect to the stator windings. It induces currents in the rotor windings, and torque is produced in the direction of the revolving field. The slip of the motor will automatically adjust itself, until the internal torque, which is proportional to the product of the stator and rotor currents, is equal to the sum of the load torque and the torque due to the losses and thus sufficient to keep the motor operating. If the motor is incapable of producing an internal torque of the necessary magnitude, it will "break down" and the rotor will come to rest.

When the motor draws unbalanced currents equivalent to two balanced current systems of opposite phase sequence, its performance may be determined by *considering the effect of each system separately and superimposing the results.*[1] Each of the balanced current systems may be looked upon as giving rise to a

[1] Lyon, W. V., "Unbalanced Three-phase Circuits," *Elec. World*, p. 1304, 1920.

Slepian, J., "Induction Motors on Unbalanced Voltages," *Elec. World*, p. 313, 1920.

Dudley, A. M., "Induction Motors on Unbalanced Circuits. Vector Method of Analysis of Unsymmetrical Systems," *Elec. Jour.*, p. 339, 1924.

Schoenfeld, O. C., "Effect of Unbalanced Voltages on the Operation of Induction Motors," *Elec. Jour.*, p. 30, 1925.

revolving field in the air gap. Obviously, these fields, *viz.*, the positive- and the negative-sequence fields, will *rotate in opposite directions* and will, hence, in conjunction with the currents they induce in the rotor windings, *produce oppositely acting torques.* The positive-sequence torque acts in the direction of rotation of the rotor. This torque, therefore, in order to make operation possible, must equal the sum of the negative-sequence torque and the torques due to the load and the losses.

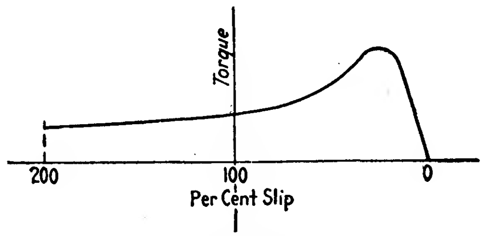

FIG. 80.—Slip-torque curve of a polyphase induction motor operating on balanced voltages.

When the rotor turns over at synchronous speed, the positive-sequence slip, which is the actual slip of the motor, is zero, while the negative-sequence slip has its maximum value, namely, 2 or 200 per cent. If the slip (*i.e.*, the positive-sequence slip) is s, the negative-sequences lip is evidently $2 - s$. Figure 80 shows the speed-torque or slip-torque curve of a polyphase induction motor

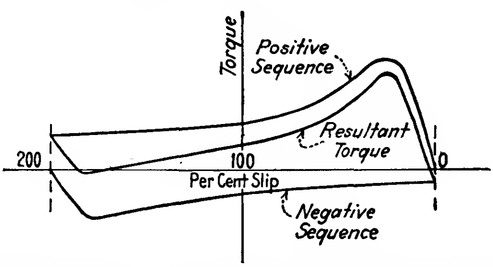

FIG. 81.—Positive-sequence, negative-sequence, and resultant slip-torque curves of a polyphase induction motor operating on unbalanced voltages.

in the region from 0 to 200 per cent slip for operation on constant balanced voltages. The range between 100 per cent slip (standstill) and 200 per cent slip is obtained by driving the rotor in the reverse direction by external means. Assuming that the motor operates on unbalanced voltages and that slip-torque curves, as in Fig. 80, are available for values of balanced voltages equal to

the positive- and negative-sequence components of the unbalanced system actually impressed, the complete slip-torque characteristic for the unbalanced operation may be obtained by superposition, as shown in Fig. 81. This shows that unbalance reduces the net torque for all values of slip. Hence, in order to deliver a specified pulley torque, the slip must increase when the impressed voltages become unbalanced. The starting torque, as well as the pull-out torque, is lower than for balanced operation. The slip at pull-out, however, remains sensibly the same.

Figure 82 shows the slip-current and slip-power-factor relations for positive- and negative-sequence systems. From the values of

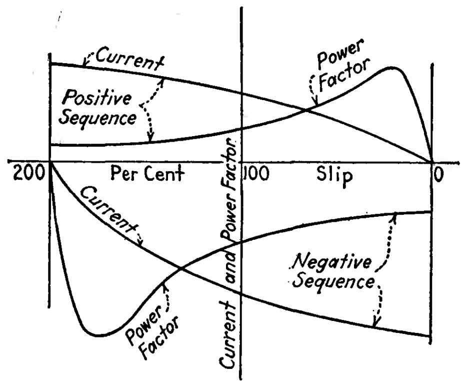

FIG. 82.—Positive- and negative-sequence current and power-factor curves plotted *versus* slip of a polyphase induction motor operating on unbalanced voltages.

current and power factor, the power due to positive- and negative-sequence components may readily be calculated, and the net power determined for any slip. In order to determine the unbalanced currents, the positive- and negative-sequence currents must be combined vectorially in the proper phase relation. Assuming that the relative phase position of the component voltages, as well as their magnitudes, is known, the phase position of the currents is immediately ascertained by means of the power-factor angles, as indicated in the vector diagrams (Fig. 83). The combination of the positive- and negative-sequence currents may evidently be carried through either analytically or graphically.

With unbalanced currents, the copper losses and, hence, the amounts of heat developed in the three phases of the stator will be unequal. One phase (and frequently two, depending on the amount of unbalance) will be heated considerably more than

when the motor delivers the same output on balanced voltages. When the unbalanced currents are determined, the copper loss in each stator phase can readily be calculated.

The copper losses in the rotor phases, although higher than for balanced operation, will be identical even when the motor operates on unbalanced voltages. This is due to the fact that each rotor phase carries the same effective values of two currents of *different frequency,* in magnitude equal to the positive- and negative-sequence load components in the stator windings. The copper loss, therefore, in each rotor phase is obviously obtained by calculating, separately, and adding the copper losses due to the positive- and negative-sequence rotor currents. Care should, of course, be taken that the correct rotor resistance is used for each frequency.

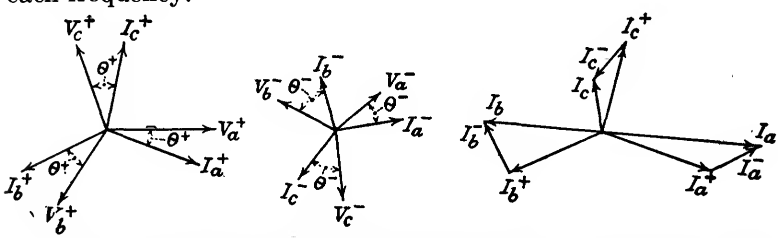

FIG. 83.—Vector diagrams of voltages and currents in a polyphase induction motor operating on unbalanced voltages.

The aggregate copper loss may be computed directly from the positive- and negative-sequence currents. Neglecting exciting current, it is given by

$$P_{Cu} = 3[(I^+)^2 r_e^+ + (I^-)^2 r_e^-] \tag{1}$$

where r_e^+ and r_e^- are the equivalent resistances per phase for the positive- and negative-sequence systems, respectively. The total copper loss is larger for unbalanced than for balanced operation, assuming the same power output in the two cases.

The core loss will usually increase somewhat when the motor is operated on unbalanced voltages. The increase is principally due to the introduction of a double-frequency core loss in the rotor, which, under balanced operation, is practically without core loss on account of the low frequency of the rotor flux. A part of the increase in total core loss may also be due to field distortion caused by the unbalance.

It is seen that both the total core loss and the total copper loss increase when the impressed voltages are unbalanced. The

friction and windage loss will remain sensibly unaltered, since the reduction in speed for the same output is not very large. The motor will consequently suffer a decrease in efficiency when operated on unbalanced voltages.

Three-phase Induction Motor on Unbalanced Voltages. Analytical Treatment.—The following notation will be used in the discussion below:

V_1 = impressed voltage per phase
I_1, I_2 = stator and rotor current per phase
I_e = exciting (no-load) current
T_2 = internal torque per phase
P_2 = internal power per phase
p = number of poles
f_1 = impressed frequency
s = slip
r_1 = stator effective resistance per phase at impressed frequency
r_2 = rotor resistance per phase (effective or ohmic)
r_2' = rotor effective resistance per phase at double impressed frequency
x_1, x_2 = stator and rotor reactance per phase at impressed frequency
$Z_1 = r_1 + jx_1$ = stator impedance
Z_e = equivalent impedance corresponding to slip s.

In subsequent formulas, all quantities are reduced to stator. Note that magnitude *as well as phase* of the rotor currents (I_2) are referred to stator.

When the induction motor operates on balanced voltages, its performance may be calculated by the following equations:[1]

$$V_1' = V_1 - I_eZ_1 \tag{2}$$

$$V_1' = I_2Z_e = I_2\left[r_1 + \frac{r_2}{s} + j(x_1 + x_2)\right] \tag{3}$$

$$P_2 = I_2^2r_2\frac{1-s}{s} = \frac{(V_1')^2(1-s)sr_2}{(sr_1 + r_2)^2 + s^2(x_1 + x_2)^2} = \frac{(V_1')^2(1-s)\frac{r_2}{s}}{\left(r_1 + \frac{r_2}{s}\right)^2 + (x_1 + x_2)^2} \tag{4}$$

[1] LAWRENCE, R. R., "Principles of Alternating-current Machinery," McGraw-Hill Book Company, Inc., New York, 1916 and 1920.

$$T_2 = \frac{p}{4\pi f_1} I_2^2 \frac{r_2}{s} = \frac{p(V_1')^2}{4\pi f_1} \frac{sr_2}{(sr_1 + r_2)^2 + s^2(x_1 + x_2)^2}$$

$$= \frac{p(V_1')^2}{4\pi f_1} \frac{\frac{r_2}{s}}{\left(r_1 + \frac{r_2}{s}\right)^2 + (x_1 + x_2)^2} \tag{5}$$

Equation (2) gives the voltage which is effective in producing the load component of the current (rotor current) by subtracting the stator impedance drop due to the exciting current from the impressed voltage. Since the exciting current always lags the impressed voltage by a large angle, it is usually sufficiently accurate to perform this subtraction algebraically instead of vectorially. Equation (3) relates the voltage V_1', the rotor current, and the equivalent impedance of the motor. Equations (4) and (5) give the internal power and torque, respectively, in terms of the rotor current or the voltage V_1', the slip, and the constants of the motor.

Since holding, in general, for balanced voltages and currents, the equations above (equations (2) to (5) inclusive) hold separately for positive- and negative-sequence systems. Hence, when the motor is operating on an unbalanced circuit and either the component voltages or currents are known, the performance of the motor may be calculated by means of these equations.

For the positive-sequence system, the slip equals the actual slip s, while the slip for the negative-sequence system is $2 - s$. The stator resistance is the same for both systems, namely, the effective resistance at impressed frequency. The rotor resistance, however, is different for the two systems. For the positive, the ohmic rotor resistance; and for the negative, the effective rotor resistance at double impressed frequency should be used. This, of course, assumes operation at a small slip. The equivalent reactance $(x_1 + x_2)$ is usually considered the same for the two systems and taken equal to the reactance at impressed frequency, as determined by a test with the rotor blocked. Strictly, the negative-sequence rotor reactance may be slightly less than its positive-sequence reactance, due to a small decrease in inductance caused by skin effect. This, however, will be neglected.

The exciting currents may be taken into account for both the positive- and the negative-sequence system, for the positive-sequence system alone, or they may be completely ignored. Since the positive-sequence exciting current is quite appreciable

(20 to 30 per cent of the full-load current) it is advisable to take this into consideration. It is hardly worth while, however, to take the negative-sequence exciting current into account, *since it is always very small.* This is due to the fact that the negative-sequence equivalent impedance for values of slip in the stable operating range is very small and seldom exceeds 15 per cent of the equivalent impedance for the positive-sequence system. This, in connection with the fact that the impressed negative-sequence voltage is usually much smaller than the positive-sequence voltage, causes the induced voltage of negative sequence to be quite low. Consequently, the negative-sequence flux is small; and the excitation required to maintain it, negligible.

Assuming that the motor operates on an unbalanced system of voltages whose positive- and negative-sequence components are V_1^+ and V_1^-, respectively, and neglecting negative-sequence excitation, the calculation may be carried through as follows:

$$I_2^+ = \frac{V_1^+ - I_e^+ Z_1}{r_1 + \frac{r_2}{s} + j(x_1 + x_2)} \tag{6}$$

$$I_2^- = \frac{V_1^-}{r_1 + \frac{r_2'}{2 - s} + j(x_1 + x_2)} \tag{7}$$

$$P_2^+ = (I_2^+)^2 r_2 \frac{1 - s}{s} \tag{8}$$

$$P_2^- = -(I_2^-)^2 r_2' \frac{1 - s}{2 - s} \tag{9}$$

$$T_2^+ = \frac{p}{4\pi f_1}(I_2^+)^2 \frac{r_2}{s} \tag{10}$$

$$T_2^- = -\frac{p}{4\pi f_1}(I_2^-)^2 \frac{r_2'}{2 - s} \tag{11}$$

The total internal power and torque are now given by

$$P_0 = 3(P_2^+ + P_2^-) \tag{12}$$

$$T_0 = 3(T_2^+ + T_2^-) \tag{13}$$

The unbalanced stator currents are

$$I_{1a} = I_{1a}^+ + I_{1a}^- = I_{ea}^+ + I_{2a}^+ + I_{2a}^- \tag{14}$$

$$I_{1b} = I_{1b}^+ + I_{1b}^- = I_{eb}^+ + I_{2b}^+ + I_{2b}^- \tag{15}$$

$$I_{1c} = I_{1c}^+ + I_{1c}^- = I_{ec}^+ + I_{2c}^+ + I_{2c}^- \tag{16}$$

and the total copper loss becomes

$$P_{Cu} = 3[(I_e^+ + I_2^+)^2 r_1 + (I_2^+)^2 r_2 + (I_2^-)^2 (r_1 + r_2')] \tag{17}$$

Balancing Effect of a Three-phase Induction Motor.—Consider a generating station supplying power over a feeder circuit to an

unbalanced three-phase load. The currents, as well as the voltages at the load, will then, in general, be unbalanced. If the capacity of the generating station is large compared to the load, the unbalance in the internal impedance drops of the generators will be inappreciable, and the station voltages remain very nearly balanced, in spite of the unbalanced load.[1]

If a three-phase induction motor, loaded or unloaded, is connected to the system at the unbalanced load, it *will tend to restore the balance of the system.* In order to obtain this balancing effect, it is necessary that there be impedance between the motor and the point at which balanced voltages are maintained. The

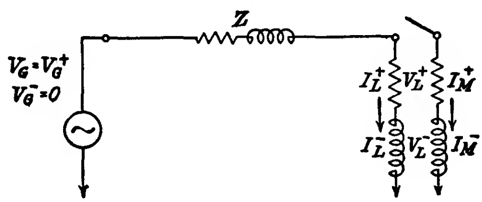

FIG. 84.—Single-line diagram of a three-phase feeder circuit supplying an unbalanced load. A polyphase induction motor is floated on the system in parallel with the load.

induction motor will draw negative-sequence currents which will add to the negative-sequence currents taken by the load in such a manner that the resultant negative-sequence current carried by the line is reduced. As a consequence of this, the negative-sequence line drops decrease, and an improvement in the voltage balance at the load is effected.

Figure 84 shows one phase of the system. There is no neutral, and it is assumed that the impedances of the three conductors are equal and independent of the division of current among them. Before the induction motor is added, the positive- and negative-sequence voltages at the load are given by

$$V_{L1}^+ = V_G^+ - I_{L1}^+ Z \tag{18}$$
$$V_{L1}^- = -I_{L1}^- Z \tag{19}$$

The degree of voltage unbalance at the load is, thus,

$$\frac{V_{L1}^-}{V_{L1}^+} = \frac{-I_{L1}^- Z}{V_G^+ - I_{L1}^+ Z} \tag{20}$$

[1] See paper by W. V. LYON, *loc. cit.*

The motor is now connected to the circuit. If it is assumed that the currents taken by the unbalanced load do not change when the motor is added, the voltages at the load now become

$$V_{L2}^+ = V_G^+ - (I_{L1}^+ + I_M^+)Z = I_M^+ Z_M^+ \tag{21}$$

$$V_{L2}^- = -(I_{L1}^- + I_M^-)Z = I_M^- Z_M^- \tag{22}$$

where Z_M^+ and Z_M^- represent the positive- and negative-sequence impedances of the induction motor, respectively. The ratio of the positive- and the negative-sequence voltage at the load after and before the induction motor is added is now very simply given by

$$\frac{V_{L2}^+}{V_{L1}^+} = \frac{Z_M^+}{Z + Z_M^+} \tag{23}$$

$$\frac{V_{L2}^-}{V_{L1}^-} = \frac{Z_M^-}{Z + Z_M^-} \tag{24}$$

Equations (23) and (24) show that there is a reduction in both the positive- and the negative-sequence voltage at the load when the induction motor is added. The new degree of unbalance is most conveniently obtained by combining equations (23) and (24). Thus,

$$\frac{V_{L2}^-}{V_{L2}^+} = \frac{V_{L1}^-}{V_{L1}^+} \cdot \frac{Z\,Z_M^- + Z_M^+ Z_M^-}{Z\,Z_M^+ + Z_M^+ Z_M^-} \tag{25}$$

This equation shows that there will be a reduction in the degree of voltage unbalance and, hence, a tendency to balance up the circuit whenever $Z_M^- < Z_M^+$. The induction motor, as previously brought out, has a negative-sequence impedance which is considerably less than its positive-sequence impedance and will, hence, exert a pronounced balancing effect. It is evident, and this may appropriately be emphasized at this point, that any piece of apparatus whose impedance to negative-sequence currents is smaller than to positive-sequence currents may be used for the purpose of balancing up a circuit. The machine does not necessarily have to be an induction motor. A synchronous motor for instance, may be used.

From equation (22) is obtained

$$\frac{I_M^-}{I_{L1}^-} = -\frac{Z}{Z + Z_M^-} \tag{26}$$

This equation shows that if Z_M^- were zero, the negative-sequence currents taken by the load and the induction motor would be exactly equal and opposite and, hence, cancel on the line. Equation (25) shows that for $Z_M^- = 0$ the degree of voltage unbalance

is also zero. Balance would be completely restored. Hence the smaller the negative-sequence impedance of the motor is the more effective it is as a balancing agent.

EXAMPLE 1

Statement of Problem

Power is supplied from a 60-cycle station over a three-phase, high-tension feeder to an unbalanced load. The station instruments are connected in the circuit at the terminals of the generator. The resistance and reactance of the line and transformers are 0.5 and 0.6 ohm, respectively, per conductor, referred to the low-tension side.

On open circuit, the station voltmeter indicates 7,200 volts, while that on the low-tension side of the load transformers shows, 2,400 volts. With the unbalanced load connected to the feeder, the station voltmeter shows 7,200 volts; and the line ammeters, 80, 50, and 90 amp.; the polyphase wattmeter indicates 698 kw. Assume that the station voltages are balanced and that the line currents taken by the load do not change when a 1,000-hp., 2,200-volt induction motor is floated across the feeder at the load. Assume that the slip is zero.

The induction motor has the following constants per phase, assuming Y-connection: $r_1 = 0.10$ ohm at 60 cycles, $r_2 = 0.08$ ohm at 0 cycles, $r_2 = 0.15$ ohm at 120 cycles, $x_1 = x_2 = 0.30$ ohm. The no-load line current is 75 amp. at 0.1 power factor and rated voltage.

1. What are the degrees of voltage unbalance at the load before and after the induction motor is added?

2. What is the ratio of the negative-sequence currents carried by the feeder before and after the induction motor is added?

Solution

1. Overall ratio of transformation $= \dfrac{7{,}200}{2{,}400} = 3.$

The positive- and negative-sequence components of the unbalanced currents are determined graphically in the diagram, Fig. 85. Only the magnitude of the components is required.

a. Unbalanced Load on Line:

Generator voltages to neutral referred to low-tension side. $\left\{\begin{array}{l} V_G^+ = \dfrac{7{,}200}{3 \times \sqrt{3}} = 1{,}386 \text{ volts} \\ V_G^- = 0 \end{array}\right.$

Load currents from diagram referred to low-tension side. $\left\{\begin{array}{l} I_L^+ = 71.4 \times 3 = 214.2 \text{ amp.} \\ I_L^- = 23.1 \times 3 = 69.3 \text{ amp.} \end{array}\right.$

Power factor at generator (positive sequence)

$$\cos\theta_G = \frac{698{,}000}{\sqrt{3} \times 7{,}200 \times 71.4} = 0.785 \qquad \sin\theta_G = 0.618$$

Phase voltages at the load

$$V_{L1}^+ = V_G^+ - I_L^+Z = 1{,}386 - 214.2(0.785 - j0.618)(0.5 + j0.6)$$
$$= 1{,}222 - j34.8 = 1{,}222 \text{ volts}$$
$$V_{L1}^- = -I_{L1}^-Z = -69.3(0.5 + j0.6) = -34.7 - j41.6 = 54.1 \text{ volts}$$

Degree of voltage unbalance at the load

$$\frac{V_{L1}^-}{V_{L1}^+}100 = \frac{54.1 \times 100}{1{,}222} = 4.43 \text{ per cent}$$

b. Unbalanced Load and Three-phase Induction Motor on Line.—Figure 86 is a vector diagram of the positive-sequence quantities after the induction

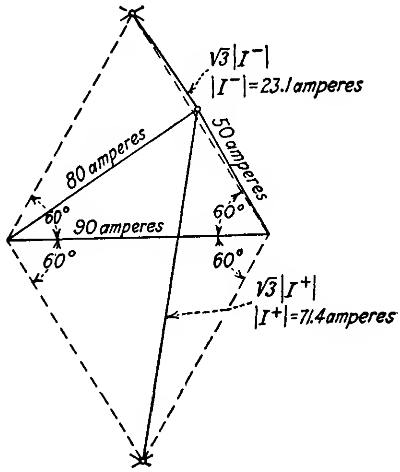

Fig. 85.—Graphical determination of positive- and negative-sequence currents (Example 1).

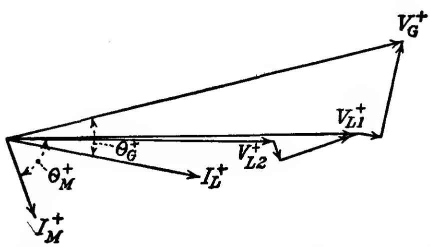

Fig. 86.—Vector diagram of positive-sequence voltages and currents (Example 1). Note that the angle marked Θ_G^+ corresponds to the power-factor angle at the generator before the induction motor is added.

motor is added. It is assumed that the currents taken by the unbalanced load remain the same as before.

$$V_{L1}^+ = V_{L2}^+ + I_M^+Z \qquad (a)$$

$$\cos\theta_M^+ = 0.1 \qquad \sin\theta_M^+ = 0.995$$

Using V_{L2}^+ as reference axis and inserting numerical values, equation (a) may be written

$$1{,}222\ \underline{/a} = V_{L2}^+ + 75(0.1 - j0.995)(0.5 + j0.6)$$
$$= V_{L2}^+ + 48.5 + j33.0$$

Hence,

$$1{,}222^2 = (V_{L2}^+ + 48.5)^2 + 33.0^2$$

which, solved for V_{L2}^{+}, gives

$$V_{L2}^{+} = 1{,}173 \text{ volts}$$

$$V_{L2}^{-} = V_{L1}^{-} \frac{Z_M^{-}}{Z + Z_M^{-}} \tag{b}$$

Negative-sequence impedance of motor at zero slip

$$Z_M^{-} = r_1 + \frac{r_2'}{2 - s} + j(x_1 + x_2) = 0.1 + \frac{0.15}{2} + j0.6$$

$$= 0.175 + j0.6 = 0.625 \text{ ohm}$$

Substituting in equation (b) gives

$$V_{L2}^{-} = 54.1 \frac{0.175 + j0.6}{0.5 + j0.6 + 0.175 + j0.6} = 54.1 \frac{0.175 + j0.6}{0.675 + j1.2}$$

$$= 54.1 \frac{0.625}{1.376} = 24.6 \text{ volts}$$

Degree of voltage unbalance at the load

$$\frac{V_{L2}^{-}}{V_{L2}^{+}}100 = \frac{24.6 \times 100}{1{,}173} = 2.10 \text{ per cent}$$

Hence, the floating of the induction motor on the circuit in parallel with the unbalanced load reduces the degree of voltage unbalance from 4.43 to 2.10 per cent.

2. The ratio of the negative-sequence currents carried by the feeder before and after the induction motor is added is given by

$$\frac{I_2^{-}}{I_L^{-}} = 1 - \frac{Z}{Z + Z_M^{-}} \tag{c}$$

$$\frac{I_2^{-}}{I_L^{-}} = 1 - \frac{0.5 + j0.6}{0.675 + j1.2} = 0.442 + j0.103 = 0.454\underline{/13^\circ.1}$$

The ratio in per cent is, therefore,

$$\frac{I_2^{-}}{I_L^{-}}100 = 45.4 \text{ per cent}$$

Resumé

The calculations show that the floating of the induction motor on the circuit in parallel with the unbalanced load more than halves the negative-sequence currents carried by the feeder and thus reduces the degree of voltage unbalance at the load from 4.43 to 2.10 per cent. Hence, material improvement in the conditions is realized.

The calculated results, however, should not be considered exact, since they are based on the assumption that the load currents remained the same after the induction motor was added. If greater accuracy is desired, the change in the load currents, which, of necessity, must be caused by the connection of the induction motor, should be taken into account.

Single-phase Operation of a Three-phase Induction Motor.—An induction motor operating on a three-phase circuit will continue to operate on single-phase supply if one of its phases is opened. This statement is based on the assumption that the torque of the load which the motor has to carry does not exceed the breakdown torque for single-phase operation. The opening

of one phase will be accompanied by an increase in slip, line current, and losses and a decrease in efficiency.

This last statement assumes that the three-phase voltages are balanced or, at least, nearly balanced. If the applied voltages during three-phase operation are badly unbalanced, the continuous capacity of the motor may become so severely reduced that it actually may be advantageous to open one phase and allow the motor to run single phase. The continuous capacity of a polyphase induction motor operating single phase is somewhat less than 70 per cent of its rated capacity.[1]

During single-phase operation, the positive- and negative-sequence voltages across the stator terminals change with the

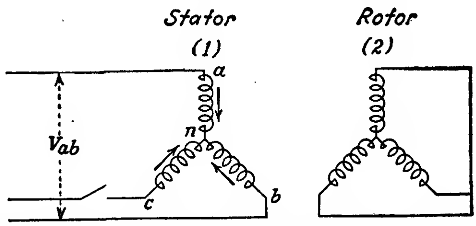

FIG. 87.—Circuit diagram of a three-phase Y-connected induction motor operating single-phase.

slip. They are not fixed by the applied line voltage as in the unbalanced three-phase case. In the single-phase case, on the other hand, there is a definite relation between the positive- and negative-sequence currents. As a matter of fact, these components are all the time equal in magnitude and in a fixed phase position relative to each other.

Consider the three-phase induction motor shown in Fig. 87. Power is supplied to the terminals a and b only. The equations for the stator currents are

$$I_{1a} = I_{1a}^{+} + I_{1a}^{-} \tag{27}$$

$$I_{1b} = I_{1a}^{+}\overline{\backslash 120^\circ} + I_{11a}^{-}\underline{/120^\circ} \tag{28}$$

$$I_{1c} = I_{1a}^{+}\underline{/120^\circ} + I_{1a}^{-}\overline{\backslash 120^\circ} \tag{29}$$

$$I_{1a} + I_{1b} + I_{1c} = 0 \tag{30}$$

$$I_{1c} = 0 \tag{31}$$

[1] See paper by J. SLEPIAN, *loc. cit.* It is stated in this paper that, if the degree of unbalance (unbalance factor) of the three-phase line exceeds 70 per cent of the *percentage reactance* of the motor, the output is reduced to less than 70 per cent of normal output. Hence, better continuous performance will be obtained by opening one phase of the motor.

These equations are satisfied by the system of currents shown in Fig. 88.

Obviously, the numerical value of the components is equal to the actual current taken divided by $\sqrt{3}$, *viz.*,

$$|I_1^+| = |I_1^-| = \frac{|I_1|}{\sqrt{3}} \tag{32}$$

In deriving the formulas required for the calculation of single-phase performance, the positive-sequence exciting current may or may not be taken into account. As in the unbalanced three-phase case, the current necessary for negative-sequence excitation may safely be ignored. It will always be very small. The equations so far established for the stator currents are valid, whether the exciting currents are considered or omitted.

Analysis of Performance, Neglecting Exciting Currents.—When the exciting currents are neglected, the total positive-sequence as well as the total negative-sequence stator currents will have equivalent components flowing in the rotor windings. The

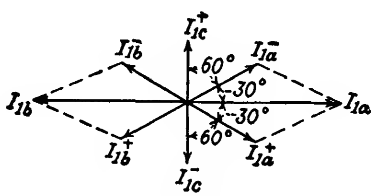

FIG. 88.—Vector diagram of stator currents of the induction motor in Fig. 87.

vector diagram, Fig. 88, may, therefore, be looked upon as representing also the rotor currents referred to stator. The values of the components as determined by equation (32) may consequently be used in the general three-phase equations for power and torque.

The relation between the impressed single-phase voltage and the current taken by the motor is obtained by expressing the terminal voltages across the two active phases in terms of their positive- and negative-sequence components. Thus,

$$V_{1a} = V_{1a}^+ + V_{1a}^- = I_{1a}^+ Z^+ + I_{1a}^- Z^- \tag{33}$$

$$V_{1b} = V_{1b}^+ + V_{1b}^- = I_{1b}^+ Z^+ + I_{1b}^- Z^- \tag{34}$$

Subtracting equation (34) from equation (33) gives

$$\begin{aligned} V_{ab} &= V_{1a} - V_{1b} = (I_{1a}^+ - I_{1b}^+)Z^+ + (I_{1a}^- - I_{1b}^-)Z^- \\ &= I_{1a}Z^+ + I_{1a}Z^- = I_{1a}(Z^+ + Z^-) \end{aligned} \tag{35}$$

Hence,

$$I_{1a} = \frac{V_{ab}}{Z^+ + Z^-} = \frac{V_{ab}}{r_1 + \frac{r_2}{s} + j(x_1+x_2) + r_1 + \frac{r_2'}{2-s} + j(x_1 + x_2)}$$

$$= \frac{V_{ab}}{2r_1 + \frac{r_2}{s} + \frac{r_2'}{2-s} + j2(x_1 + x_2)} \tag{36}$$

Making use of equations (8) to (13) inclusive and dropping subscripts referring to phase, the total internal power and torque become

$$P_0 = 3\left[(I_1^+)^2 r_2 \frac{1-s}{s} - (I_1^-)^2 r_2' \frac{1-s}{2-s}\right]$$

$$= I_1^2(1-s)\left[\frac{r_2}{s} - \frac{r_2'}{2-s}\right] \tag{37}$$

$$T_0 = \frac{3p}{4\pi f_1}\left[(I_1^+)^2 \frac{r_2}{s} - (I_1^-)^2 \frac{r_2'}{2-s}\right]$$

$$= \frac{p}{4\pi f_1} I_1^2 \left[\frac{r_2}{s} - \frac{r_2'}{2-s}\right] \tag{38}$$

Since $\frac{r_2'}{2-s}$ is very much smaller than $\frac{r_2}{s}$ for ordinary values of slip, the effect of the negative-sequence power and torque is small. The reduction in the total power and torque suffered during single-phase operation (assuming the same slip as for three-phase operation) is principally due to the decrease of the positive-sequence power and torque. This is due to the fact that the positive-sequence current, when operated single phase, is less than the current taken during three-phase operation on balanced voltages of the same magnitude as the single-phase voltage. This is readily seen, since, quite obviously,

$$\frac{V_{ab}}{\sqrt{3}\,(Z^+ + Z^-)} < \frac{V_{ab}}{\sqrt{3}Z^+} \tag{39}$$

The total copper loss during single-phase operation when exciting currents are ignored becomes

$$P_{Cu} = 3[(I_1^+)^2(r_1 + r_2) + (I_1^-)^2(r_1 + r_2')] \tag{40}$$
$$= I_1^2(2r_1 + r_2 + r_2')$$

When a Y-connected induction motor operates single phase, as shown in Fig. 87, voltages will appear from line to neutral on the two active phases, as given by equations (33) and (34). In general, a voltage will also appear across the idle phase. The value of these voltages depends on the slip. Neglecting exciting

currents, they are readily expressed in terms of the voltage V_{ab} impressed on the active phases. Thus,

$$V_a = \frac{V_{ab}\backslash 30^\circ}{\sqrt{3}} \cdot \frac{Z^+ + Z^-/60^\circ}{Z^+ + Z^-} \tag{41}$$

$$V_b = \frac{V_{ab}\backslash 150^\circ}{\sqrt{3}} \cdot \frac{Z^+ + Z^-\backslash 60^\circ}{Z^+ + Z^-} \tag{42}$$

$$V_c = \frac{V_{ab}/90^\circ}{\sqrt{3}} \cdot \frac{Z^+ - Z^-}{Z^+ + Z^-} \tag{43}$$

When the motor runs at synchronous speed, the three phase voltages form a balanced system. At synchronous speed, the positive-sequence impedance becomes infinite ($Z^+ = \infty$), and equations (41), (42), and (43) reduce to

$$V_a = \frac{V_{ab}\backslash 30^\circ}{\sqrt{3}} \tag{44}$$

$$V_b = \frac{V_{ab}\backslash 150^\circ}{\sqrt{3}} \tag{45}$$

$$V_c = \frac{V_{ab}/90^\circ}{\sqrt{3}} \tag{46}$$

Fully balanced voltages, as indicated above, will never be obtained in actual practice, even if the motor would operate at exactly synchronous speed. This is due to the fact that, even at synchronous speed, the motor must draw a certain single-phase current, part of which will obviously be of negative sequence and, hence, introduce a negative-sequence component in the voltages to neutral.

When the motor is at standstill, the slip is unity, and the positive- and negative-sequence impedances are equal. For this condition, the phase voltages become

$$V_a = \frac{V_{ab}}{2} \tag{47}$$

$$V_b = -\frac{V_{ab}}{2} \tag{48}$$

$$V_c = 0 \tag{49}$$

As seen, exactly one-half of the applied single-phase voltage appears across each of the active phases, while the voltage across the idle phase is zero.

The voltages existing between the terminals of the active phases and the terminal of the idle phase are given by

$$V_{bc} = V_b - V_c = V_{ab}\frac{Z^+\overline{\backslash 120^\circ} + Z^-\underline{/120^\circ}}{Z^+ + Z^-} \tag{50}$$

$$V_{ca} = V_c - V_a = V_{ab}\frac{Z^+\underline{/120^\circ} + Z^-\overline{\backslash 120^\circ}}{Z^+ + Z^-} \tag{51}$$

At synchronous speed, the magnitude of the voltages becomes equal to the single-phase voltage impressed. Equations (50) and (51) reduce to

$$V_{bc} = V_{ab}\overline{\backslash 120^\circ} \tag{52}$$

$$V_{ca} = V_{ab}\underline{/120^\circ} \tag{53}$$

At standstill, these voltages become equal and are given by

$$V_{bc} = V_{ca} = -\frac{V_{ab}}{2} \tag{54}$$

EXAMPLE 2

Statement of Problem

Compare the line currents and the internal torque developed by the three-phase induction motor described in Example 1 when operated on a balanced three-phase, 2,200-volt circuit and on a single-phase 2,200-volt circuit. The slip is 2.5 per cent in each case. Neglect the no-load current.

Solution

Line current, three-phase

$$I_{3-\phi} = \frac{V/\sqrt{3}}{r_1 + \frac{r_2}{s} + j(x_1 + x_2)} \tag{a}$$

Line current, single-phase

$$I_{1-\phi} = \frac{V}{Z^+ + Z^-} = \frac{V}{2r_1 + \frac{r_2}{s} + \frac{r_2'}{2-s} + j2(x_1 + x_2)} \tag{b}$$

Torque, three-phase

$$T_{3-\phi} = \frac{3p}{4\pi f_1} I_{3-\phi}^2 \frac{r_2}{s} = k3I_{3-\phi}^2 \frac{r_2}{s} \tag{c}$$

Torque, single-phase

$$T_{1-\phi} = \frac{p}{4\pi f_1} I_{1-\phi}^2\left[\frac{r_2}{s} - \frac{r_2'}{2-s}\right] = kI_{1-\phi}^2\left[\frac{r_2}{s} - \frac{r_2'}{2-s}\right] \tag{d}$$

Inserting numerical values gives

$$I_{3-\phi} = \frac{2{,}200/\sqrt{3}}{\sqrt{\left(0.1 + \frac{0.08}{0.025}\right)^2 + 0.60^2}} = \frac{1{,}270}{3.357} = 378 \text{ amp.}$$

$$I_{1-\phi} = \frac{2{,}200}{\sqrt{\left(0.2 + \frac{0.08}{0.025} + \frac{0.15}{1.975}\right)^2 + 4 \times 0.60^2}} = \frac{2{,}200}{3.675} = 598 \text{ amp.}$$

$$T_{3-\phi} = k \times 3 \times 378^2 \times \frac{0.08}{0.025} = k \times 378^2 \times 9.6$$

$$T_{1-\phi} = k \times 598^2\left[\frac{0.08}{0.025} - \frac{0.15}{1.975}\right] = k \times 598^2 \times 3.124$$

$$\frac{\text{Three-phase current}}{\text{Single-phase current}} = \frac{378}{598} = 0.632$$

$$\frac{\text{Three-phase torque}}{\text{Single-phase torque}} = \frac{378^2 \times 9.6}{598^2 \times 3.124} = 1.225$$

Analysis of Performance, Taking Positive-sequence Exciting Current into Account.—When the total positive-sequence stator currents contain components required for excitation, the latter must be subtracted from the total values in order to give the positive-sequence system reproduced in the rotor. This is indicated in the vector diagram, Fig. 89, in which I_{1a}^+, I_{1b}^+, and

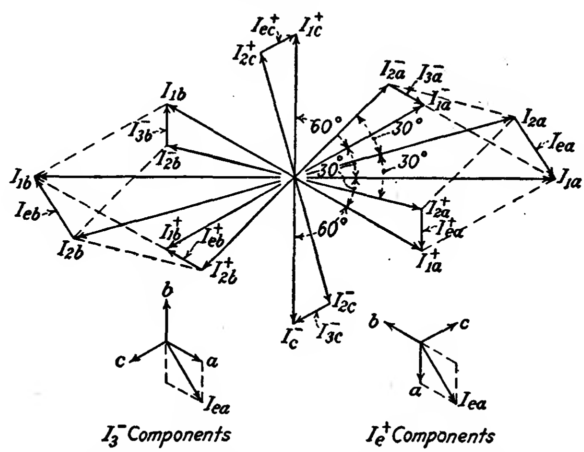

FIG. 89.—Vector diagram of stator and rotor currents of the induction motor in Fig. 87, positive-sequence exciting current being taken into account.

I_{1c}^+ represent the positive-sequence stator currents; I_{ea}^+, I_{eb}^+, and I_{ec}^+ the exciting currents; and I_{2a}^+, I_{2b}^+, and I_{2c}^+ the stator components of the positive-sequence rotor currents.

Although the total stator currents of negative sequence are reproduced in the rotor, it is convenient for the analysis to split also each current in this system into two components. Thus, in phase a, the current I_{1a}^- is decomposed into I_{2a}^- equal in magnitude to I_{2a}^+ and displaced 60 deg. from the latter, and I_{3a}^- equal to the positive-sequence exciting current and displaced 60 deg. from I_{ea}^+. The negative-sequence currents in the other phases are split up in a similar manner, as shown in the vector diagram. It

should be borne in mind that none of the negative-sequence components thus obtained are exciting currents. They all exist in both stator and rotor.

The following equations may be written for the voltages across the active stator phases:

$$V_{1a} = V_{1a}^+ + V_{1a}^- = I_{ea}^+ Z_1 + I_{2a}^+ Z^+ + (I_{2a}^- + I_{3a}^-)Z^- \tag{55}$$

$$V_{1b} = V_{1b}^+ + V_{1b}^- = I_{eb}^+ Z_1 + I_{2b}^+ Z^+ + (I_{2b}^- + I_{3b}^-)Z^- \tag{56}$$

Subtraction gives

$$\begin{aligned} V_{ab} = V_{1a} - V_{1b} &= (I_{ea}^+ - I_{eb}^+)Z_1 + (I_{2a}^+ - I_{2b}^+)Z^+ + \\ &\qquad (I_{2a}^- - I_{2b}^-)Z^- + (I_{3a}^- - I_{3b}^-)Z^- \\ &= I_{ea}(Z_1 + Z^-) + I_{2a}(Z^+ + Z^-) \end{aligned} \tag{57}$$

From this follows

$$I_{2a} = \frac{V_{ab} - I_{ea}(Z_1 + Z^-)}{Z^+ + Z^-} \tag{58}$$

Note that the current I_{2a}, as determined by equation (58), *does not represent the total rotor current* for single-phase operation. The latter is given by the sum of I_{2a} and the negative-sequence component I_{3a}^-; *i.e.*, the total rotor current equals $I_{2a} + I_{3a}^-$ (or $I_{2a} + I_{ea}^+ \underline{/60^\circ}$). I_{ea} is the current which the motor would actually draw at no load (assuming zero slip), when operated single phase. This no-load current is made up of a positive-sequence exciting current and a negative-sequence load component of equal magnitude but phase displaced 60 deg. from the former. The value of the single-phase no-load current is equal to the positive-sequence exciting current times $\sqrt{3}$.

From equation (58), the positive-sequence load component is obtained as

$$I_{2a}^+ = \frac{I_{2a}}{\sqrt{3}} \overline{\backslash 30^\circ} = \frac{V_{ab} - I_{ea}(Z_1 + Z^-)}{\sqrt{3}(Z^+ + Z^-)} \overline{\backslash 30^\circ} \tag{59}$$

or, omitting reference to phase,

$$I_2^+ = \frac{I_2}{\sqrt{3}} = \frac{\dfrac{V_{ab}}{\sqrt{3}} - I_e^+(Z_1 + Z^-)}{Z^+ + Z^-} = \frac{V_1'}{Z^+ + Z^-} \tag{60}$$

The total negative-sequence current in phase a is now determined by

$$I_{1a}^- = \frac{I_{1a}}{\sqrt{3}} \underline{/30^\circ} = \frac{I_{2a} + I_{ea}}{\sqrt{3}} \underline{/30^\circ} = (I_{2a}^+ + I_{ea}^+) \underline{/60^\circ} \tag{61}$$

Omitting reference to phase, this may be written

$$I_1^- = \frac{I_1}{\sqrt{3}} = \frac{I_2 + I_e}{\sqrt{3}} = I_2^+ + I_e^+ \tag{62}$$

Since the angle between I_2^+ and I_e^+ is large, they may, without appreciable error, be added as if they were in quadrature. This simplifies the calculation of the negative-sequence currents and gives

$$I_1^- = \sqrt{(I_2^+)^2 + (I_e^+)^2} \text{ (approximately)} \tag{63}$$

Having determined the positive- and negative-sequence load currents, the total internal power and torque are given by

$$P_0 = 3\left[(I_2^+)^2 r_2 \frac{1-s}{s} - (I_1^-)^2 r_2' \frac{1-s}{2-s}\right] \tag{64}$$

$$T_0 = \frac{3p}{4\pi f_1}\left[(I_2^+)^2 \frac{r_2}{s} - (I_1^-)^2 \frac{r_2'}{2-s}\right] \tag{65}$$

Making use of the approximate relation in equation (63), the power and torque formulas may also be written

$$P_0 = I_1^2(1-s)\left[\frac{r_2}{s} - \frac{r_2'}{2-s}\right] - I_e^2 r_2 \frac{1-s}{s} \tag{66}$$

$$T_0 = \frac{p}{4\pi f_1} I_1^2\left[\frac{r_2}{s} - \frac{r_2'}{2-s}\right] - \frac{p}{4\pi f_1} I_e^2 \frac{r_2}{s} \tag{67}$$

Comparison of these equations with equations (37) and (38) shows conclusively that the omission of exciting current from the analysis results in power and torque values which are too optimistic.

When exciting currents are taken into account, the total copper loss becomes

$$P_{Cu} = 3[(I_1^+)^2 r_1 + (I_2^+)^2 r_2 + (I_1^-)^2(r_1 + r_2')] \tag{68}$$

Again introducing the approximate relation expressed by equation (63), the total copper loss may be written

$$P_{Cu} = I_1^2(2r_1 + r_2 + r_2') - I_e^2 r_2 \tag{69}$$

EXAMPLE 3

Statement of Problem

A three-phase, 1,000-hp., 25-cycle, 2,200-volt, 12-pole, Y-connected induction motor operates on a balanced circuit and delivers rated output to a *constant-torque* load.

One of the lines feeding the motor is suddenly opened and the motor continues to operate single-phase. Determine *to engineering accuracy* the slip, the power output, the efficiency, and the line current taken and compare with the corresponding quantities for three-phase operation.

Stator effective resistance at 25 cycles............	0.075 ohm per phase
Rotor ohmic resistance referred to stator..........	0.083 ohm per phase
Rotor effective resistance at 25 cycles referred to stator....................................	0.095 ohm per phase

Rotor effective resistance at 50 cycles referred to stator	0.140 ohm per phase
Equivalent reactance at 25 cycles ($x_1 + x_2$)	0.626 ohm per phase
No-load line current at rated voltage	75.1 amp.
No-load power	15.2 kw.
Friction and windage loss	8.0 kw.

Solution

Three-phase Operation:

The internal power per phase is given by

$$P_2 = \frac{(V_1')^2(1 - s)\frac{r_2}{s}}{\left(r_1 + \frac{r_2}{s}\right)^2 + (x_1 + x_2)^2} \tag{a}$$

which may be written

$$[P_2r_1^2 + P_2(x_1 + x_2)^2 + (V_1')^2r_2]s^2 + [2P_2r_1r_2 - (V_1')^2r_2]s + P_2r_2^2 = 0 \tag{b}$$

Solving for the slip gives

$$s = -\frac{A}{2} \pm \sqrt{\left(\frac{A}{2}\right)^2 - B} \tag{c}$$

where

$$A = \frac{2P_2r_1r_2 - (V_1')^2r_2}{P_2[r_1^2 + (x_1 + x_2)^2] + (V_1')^2r_2} \tag{d}$$

$$B = \frac{P_2r_2^2}{P_2[r_1^2 + (x_1 + x_2)^2] + (V_1')^2r_2} \tag{e}$$

Assuming the leakage reactance to divide equally between stator and rotor gives

$$x_1 = x_2 = 0.313 \text{ ohm}$$

$$Z_1 = 0.075 + j0.313 = 0.314 \text{ ohm}$$

$$V_1 = \frac{2{,}200}{\sqrt{3}} = 1{,}270 \text{ volts}$$

$$V_1' = V_1 - I_eZ_1 = 1{,}270 - 75.1 \times 0.314 = 1{,}246 \text{ volts}$$

$$P_2 = \frac{746 + 8}{3} = 251.33 \text{ kw.}$$

$$A = \frac{2 \times 251{,}330 \times 0.075 \times 0.083 - 1{,}246^2 \times 0.083}{251{,}330(0.075^2 + 0.626^2) + 1{,}246^2 \times 0.083}$$

$$= \frac{3{,}125 - 128{,}800}{99{,}800 + 128{,}800} = -\frac{125{,}675}{228{,}600} = -0.549$$

$$B = \frac{251{,}330 \times 0.083^2}{228{,}600} = 0.007565$$

$$s = 0.2745 \pm \sqrt{0.0754 - 0.007565} = 0.2745 \pm 0.2602$$

$$s = 0.0143 \text{ or } 1.43 \text{ per cent}$$

$$P_2 = I_2^2\frac{1 - s}{s}r_2 \tag{f}$$

$$I_2 = \sqrt{\frac{P_2s}{(1 - s)r_2}} = \sqrt{\frac{251{,}330 \times 0.0143}{0.9857 \times 0.083}} = 209.5 \text{ amp.}$$

The phase of this current, with respect to the induced voltage, is given by

$$\alpha = \tan^{-1}\frac{sx_2}{r_2} = \tan^{-1}\frac{0.0143 \times 0.313}{0.083} = \tan^{-1} 0.054 = 3.1 \text{ deg.}$$

The power factor of the exciting current, with respect to the induced voltage, depends on the core loss.

$$\text{Core loss} = 15.2 - 8.0 - 3 \times 75.1^2 \times 0.075 \times 10^{-3} = 5.93 \text{ kw.}$$

$$I_{\text{core}} = \frac{5{,}930}{3 \times 1{,}246} = 1.59 \text{ amp.}$$

$$\cos\phi_n' = \frac{1.59}{75.1} = 0.0212 \qquad \phi_n' = 88.8 \text{ deg.}$$

$$\begin{aligned} I_1 &= I_2 + I_e = 209.5\,(\cos\alpha + j\sin\alpha) + 75.1(\cos\phi_n' + j\sin\phi_n') \\ &= 209.5(0.9986 + j0.0523) + 75.1(0.0212 + j0.9998) \\ &= 209 + j11.0 + 1.6 + j75.0 = 210.6 + j86.0 = 227.5 \text{ amp.} \end{aligned}$$

$$\begin{aligned} \text{Copper loss} &= 3(I_1^2 r_1 + I_2^2 r_2) \\ &= 3(227.5^2 \times 0.075 + 209.5^2 \times 0.083)10^{-3} = 22.57 \text{ kw.} \end{aligned}$$

Core loss	= 5.93 kw.
Friction and windage	= 8.00 kw.
Total losses	= 36.50 kw.

$$\text{Efficiency} = \frac{746 \times 100}{746 + 36.50} = 95.4 \text{ per cent.}$$

Internal torque per phase

$$T_2 = \frac{p}{4\pi f_1} I_2^2 \frac{r_2}{s} \tag{g}$$

Single-phase Operation:

$$K = T_2 \frac{4\pi f_1}{p} = (I_2^+)^2 \frac{r_2}{s} - (I_1^-)^2 \frac{r_2'}{2-s}$$

$$= \frac{(V)_1'^2}{(Z^+ + Z^-)^2}\left(\frac{r_2}{s} - \frac{r_2'}{2-s}\right) - (I_e^+)^2 \frac{r_2'}{2-s}$$

$$= \frac{(V_1')^2}{\left(2r_1 + \frac{r_2}{s} + \frac{r_2'}{2-s}\right)^2 + 4(x_1 + x_2)^2}\left(\frac{r_2}{s} - \frac{r_2'}{2-s}\right) - (I_e^+)^2 \frac{r_2'}{2-s} \tag{h}$$

Equation (*h*) assumes that the following relation is approximately correct:

$$I_1^- = \sqrt{(I_2^+)^2 + (I_e^+)^2} \tag{i}$$

Equation (*h*) cannot be solved for s explicitly. A solution may be obtained by computing K for assumed values of s and plotting the values *versus* slip. From this curve the exact slip may be obtained.

Since, however, the effect of the terms involving the reverse-phase slip $s' = 2 - s$ is comparatively small, this slip may be assumed constant for small changes in s without introducing appreciable error.

Equation (*h*) may be written

$$\left[\left\{K + (I_e^+)^2 \frac{r_2'}{s'}\right\}\left\{\left(2r_1 + \frac{r_2'}{s'}\right)^2 + 4\,(x_1 + x_2)^2\right\} + (V_1')^2 \frac{r_2'}{s'}\right]s^2 +$$

$$\left[\left(K + (I_e^+)^2 \frac{r_2'}{s'}\right)2r_2\left(2r_1 + \frac{r_2'}{s'}\right) - (V_1')^2 r_2\right]s + \left(K + (I_e^+)^2 \frac{r_2'}{s'}\right)r_2^2 = 0 \tag{j}$$

Solving for the slip gives

$$s = -\frac{C}{2} \pm \sqrt{\left(\frac{C}{2}\right)^2 - D} \tag{k}$$

Assume $s' = 2 - s = 1.984$ = constant, corresponding to a slip of 1.6 per cent. Assume, further, that the direct-phase exciting current is the same as for three-phase operation. Hence,

$$I_e^+ = 75.1 \text{ amp.}$$

$$Z_1 = 0.075 + j0.313 \text{ ohm}$$

$$Z^- = 0.075 + \frac{0.140}{1.984} + j0.626 = 0.1455 + j0.626 \text{ ohm}$$

$$Z_1 + Z^- = 0.2205 + j0.939 = 0.964 \text{ ohm}$$

$$V_1' = \frac{V_{\text{line}}}{\sqrt{3}} - I_e^+(Z_1 + Z^-) = 1{,}270 - 75.1 \times 0.964 = 1{,}198 \text{ volts}$$

$$\frac{r_2'}{s'} = \frac{r_2'}{2 - s} = \frac{0.140}{1.984} = 0.0705$$

$$2r_1 + \frac{r_2'}{s'} = 0.15 + 0.0705 = 0.2205$$

$$\left(2r_1 + \frac{r_2'}{s'}\right)^2 = 0.04865$$

$$(x_1 + x_2)^2 = 0.626^2 = 0.392$$

$$(V_1')^2\frac{r_2'}{s'} = 1{,}198^2 \times 0.0705 = 101{,}100$$

$$(V_1')^2 r_2 = 1{,}198^2 \times 0.083 = 119{,}100$$

$$K = T_2\frac{4\pi f_1}{p} = \frac{209.5^2 \times 0.083}{0.0143} = 254{,}700$$

$$K + (I_e^+)^2\frac{r_2'}{s'} = 254{,}700 + 75.1^2 \times 0.0705 = 255{,}100$$

$$C = \frac{255{,}100 \times 0.166 \times 0.2205 - 119{,}100}{255{,}100[0.04865 + 4 \times 0.392] + 101{,}100}$$

$$= \frac{9345 - 119{,}000}{412{,}400 + 101{,}100} = -\frac{109{,}655}{513{,}500} = -0.2137$$

$$D = \frac{255{,}100 \times 0.083^2}{513{,}500} = 0.00343$$

$$s = 0.10685 \pm \sqrt{0.0114 - 0.00343}$$

$$= 0.10685 \pm 0.08925$$

$$s = 0.0176 \text{ or } s = 1.76 \text{ per cent}$$

$$Z^+ = r_1 + \frac{r_2}{s} + j(x_1 + x_2) = 0.075 + \frac{0.083}{0.0176} + j0.626$$

$$= 4.795 + j0.626$$

$$Z^+ + Z^- = 4.941 + j1.252 = 5.095 \text{ ohms}$$

$$I_2^+ = \frac{V_1'}{Z^+ + Z^-} = \frac{1{,}198}{5.095} = 235 \text{ amp.}$$

$$I_{\text{line}} = \sqrt{3} \times \sqrt{235^2 + 75.1^2} = 427 \text{ amp. (approximately)}$$

$$P_2 = \frac{4\pi f_1}{p}(1 - s)T_2 = (1 - s)K$$

$$P_2 = (1 - 0.0176)254{,}700 = 250{,}100 \text{ watts}$$

$$\text{Power output} = 3 \times 250.1 - 8.0 = 742.3 \text{ kw.}$$

$$= 995 \text{ hp.}$$

The exact single-phase core loss is not known, and there is no data from which it can be accurately computed. It will be assumed, however, that the core loss increases by 50 per cent when the motor operates single-phase. It will be noted below that, since the core loss in this case makes up only about 11.5 per cent of the total losses, some error in the core loss is not serious unless the efficiency is desired with a great deal of precision.

$$
\begin{aligned}
\text{Copper loss} &= I_1^2(2r_1 + r_2 + r_2') - I_c^2 r_2 \\
&= [427^2(2 \times 0.075 + 0.083 + 0.14) - \\
&\quad 3 \times 75.1^2 \times 0.083]10^{-3} = 69.40 \text{ kw.} \\
\text{Core loss} &= 5.93 \times 1.5 = 8.90 \text{ kw.} \\
\text{Friction and windage} &= 8.00 \text{ kw.} \\
\text{Total losses} &= 86.30 \text{ kw.}
\end{aligned}
$$

$$
\text{Efficiency} = \frac{742.3 \times 100}{742.3 + 86.3} = 89.5 \text{ per cent}
$$

COMPARISON OF RESULTS

Items calculated	Three-phase operation	Single-phase operation	Single-phase in per cent of three-phase
Power output	1,000 hp.	995 hp.	99.5
Line current	227.5 amp.	427 amp.	187.7
Slip	1.43 per cent	1.76 per cent	123
Efficiency	95.4 per cent	89.5 per cent	93.8

Pull-out Torque and Slip.—When a three-phase induction motor is operated single phase, its pull-out torque and slip are seriously affected. It is easy to examine what the approximate pull-out torque and slip will be.[1]

For balanced three-phase operation, the internal torque per phase is given by

$$
T_2 = \frac{p(V_1')^2}{4\pi f_1} \frac{r_2/s}{\left(r_1 + \frac{r_2}{s}\right)^2 + (x_1 + x_2)^2} \tag{70}
$$

By differentiation of T_2 with respect to s, the slip at which maximum torque occurs is found to be

$$
s_m = \frac{r_2}{\sqrt{r_1^2 + (x_1 + x_2)^2}} \tag{71}
$$

and the corresponding maximum torque becomes

$$
T_m = \frac{p(V_1')^2}{4\pi f_1} \frac{1}{2[r_1 + \sqrt{r_1^2 + (x_1 + x_2)^2}]} \tag{72}
$$

As formerly stated, the torque produced by the negative sequence system during unbalanced three-phase or single-phase operation is usually insignificant. It will be neglected here.

[1] See paper by J. SLEPIAN, *loc. cit.*

The internal torque for single-phase operation is then given by

$$T_2 = \frac{p(V_1')^2}{4\pi f_1}\frac{r_2/s}{(Z^+ + Z^-)^2} = \frac{p(V_1')^2}{4\pi f_1}\frac{r_2/s}{\left(2r_1 + \frac{r_2}{s} + \frac{r_2'}{2-s}\right)^2 + 4(x_1 + x_2)^2} \tag{73}$$

Neglecting the term $\frac{r_2'}{2-s}$ in the denominator of equation (73), which does not introduce any appreciable error when the slip is not large, the slip at single-phase breakdown may be determined by differentiation of this equation. The result is

$$s_m = \frac{r_2}{2\sqrt{r_1^2 + (x_1 + x_2)^2}} \tag{74}$$

The maximum internal torque per phase is given by

$$T_m = \frac{p(V_1')^2}{4\pi f_1}\frac{1}{4[r_1 + \sqrt{r_1^2 + (x_1 + x_2)^2}\,]} \tag{75}$$

Hence, when operated single-phase, a three-phase induction motor pulls out at approximately half the slip and develops approximately half the pull-out torque corresponding to balanced three-phase operation.

The Single-phase Induction Motor.—Throughout the preceding discussion of single-phase operation, the induction motor in question was assumed to be a three-phase machine and it was shown how the analysis of its performance could be carried through with great ease by resorting to symmetrical coordinates. If the induction motor is *actually a single-phase machine,* however, its performance can also be calculated by the same general scheme. It should be carefully noted, however, that the equations previously derived are directly applicable only when the single-phase induction motor has a symmetrical polyphase rotor winding, which usually is the case.

In such a motor, the single-phase stator winding will set up a pulsating field when it carries current. This pulsating field may be broken up into two oppositely revolving fields of the very same type that would be produced by a three-phase (or any polyphase) winding. Hence, the single-phase stator winding of the motor may be assumed replaced by an equivalent three-phase winding in which positive- and negative-sequence currents flow and thus produce the requisite fields. In this manner, the single-phase induction motor problem reduces to the very same problem which

has just been treated in detail, namely, that of the three-phase induction motor operated single-phase. When the constants of the equivalent three-phase windings are known, the performance of the single-phase motor may be completely determined by methods and formulas previously given.

Figure 90 is a schematic diagram of the actual circuit of a single-phase induction motor and its three-phase equivalent. The constants per phase of the equivalent windings can be determined by a blocked test just as readily as if the motor actually were a three-phase machine.

When the rotor is blocked the slip is unity for both the positive- and negative-sequence systems. Neglecting exciting currents,

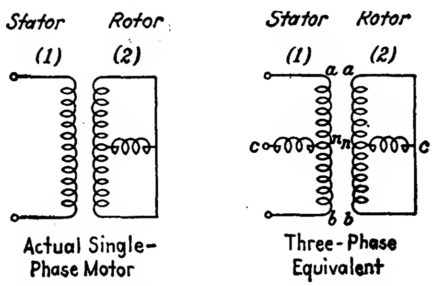

Fig. 90.—Schematic diagram of an actual single-phase induction motor and its three-phase equivalent.

the relation between the impressed voltage and the current taken is given by

$$\begin{aligned} V_{ab} &= I_{1a}(Z^+ + Z^-) = I_{1a}[2r_1 + 2r_2 + j2(x_1 + x_2)] \qquad (76) \\ &= 2I_{1a}(r_e + jx_e) = 2I_{1a}Z_e \end{aligned}$$

Here r_e and x_e are the equivalent resistance and reactance, respectively, per phase of the equivalent three-phase windings corresponding to impressed frequency.

By measuring the impressed voltage, line current, and power supplied to the single-phase motor, the equivalent constants can be calculated. Omitting subscripts referring to windings and phase, they are given by

$$Z_e = \frac{V}{2I} \qquad (77)$$

$$r_e = \frac{P}{2I^2} \qquad (78)$$

$$x_e = \sqrt{Z_e^2 - r_e^2} \qquad (79)$$

The equivalent resistance r_e is split between stator and rotor, giving the effective resistances at impressed frequency r_1 and r_2. The ohmic rotor resistance r_2, as well as its effective resistance at double impressed frequency r_2', are now determined or estimated.

EXAMPLE 4

Statement of Problem

The following data hold for a single-phase induction motor:

Rating			30 hp.
Voltage			2,200 volts
Frequency			60 cycles
Number of poles			6
No load	Voltage		2,200 volts
	Current		7.36 amp.
	Power		1,875 watts
Blocked	Voltage		620 volts
	Current		21.6 amp.
	Power		5,300 watts
Friction and windage			900 watts
Ohmic resistance between stator terminals			6.307 ohms
Ratio of effective to ohmic resistance	60 cycles	Stator	1.1
		Rotor	1.2
	120 cycles	Rotor	1.8

Stator reactance = rotor reactance when referred to same side.

Determine the pulley torque, the output, the power factor, and the efficiency when this motor operates on a 2,200-volt line at 2 per cent slip.

Solution

Equivalent three-phase constants (from blocked data)

$$z_e = \frac{V}{2I} = \frac{620}{2 \times 21.6} = 14.35 \text{ ohms}$$

$$r_e = \frac{P}{2 \times I^2} = \frac{5{,}300}{2 \times 21.6^2} = 5.68 \text{ ohms}$$

$$x_e = \sqrt{z_e^2 - r_e^2} = \sqrt{14.35^2 - 5.68^2} = 13.18 \text{ ohms}$$

$$r_1 \text{ (effective)} = \frac{6.307 \times 1.1}{2} = 3.468 \text{ ohms}$$

$$r_2 \text{ (effective)} = 5.680 - 3.468 = 2.212 \text{ ohms}$$

$$r_2 \text{ (ohmic)} = \frac{2.212}{1.2} = 1.842 \text{ ohms}$$

$$r_2' \text{ (effective)} = 1.842 \times 1.8 = 3.316 \text{ ohms}$$

At a slip s = 2 per cent

$$Z^+ + Z^- = \sqrt{\left(2r_1 + \frac{r_2}{s} + \frac{r_2'}{2-s}\right)^2 + 4(x_1 + x_2)^2}$$

$$= \sqrt{\left(2 \times 3.468 + \frac{1.842}{0.02} + \frac{3.316}{1.98}\right)^2 + 4 \times 13.18^2}$$

$$= \sqrt{10{,}143 + 694.8} = 104.1 \text{ ohms}$$

1. *Approximate Solution, Neglecting No-load Current:*

Line current

$$I = \frac{V}{Z^+ + Z^-} = \frac{2{,}200}{104.1} = 21.12 \text{ amp.}$$

Total internal torque

$$T = \frac{pI^2}{4\pi f_1}\left[\frac{r_2}{s} - \frac{r_2'}{2-s}\right]$$

$$= \frac{6 \times 21.12^2}{4\pi \times 60}\left[\frac{1.842}{0.02} - \frac{3.316}{1.98}\right]\frac{550}{746}$$

$$= \frac{6 \times 21.12^2 \times 550}{4\pi \times 60 \times 746}[92.1 - 1.674] = 236.5 \text{ ft.-lb.}$$

Friction and windage torque

$$T_{F+W} = \frac{p}{4\pi f_1(1-s)}P_{F+W}$$

$$= \frac{6 \times 900 \times 550}{4\pi \times 60 \times 0.98 \times 746} = 5.4 \text{ ft.-lb.}$$

Pulley torque $= 236.5 - 5.4 = 231.1$ ft.-lb.

Power output

$$P = \frac{4\pi f_1}{p}(1-s)T = \frac{4\pi \times 60 \times 0.98 \times 231.1}{6 \times 550} = 51.7 \text{ hp.}$$

No-load copper loss

$$2I_e^2 r_1 = 2 \times 7.36^2 \times 3.468 = 375 \text{ watts}$$

Core loss

$$P_c = 1{,}875 - 900 - 375 = 600 \text{ watts}$$

Operating at 2 per cent slip

Copper loss $= I^2(2r_1 + r_2 + r_2')$		
$= 21.12^2(2 \times 3.468 + 1.842 + 3.316)$	$=$	5,395 watts
Core loss....................................	$=$	600
Friction and windage.........................	$=$	900
Total losses	$=$	6,895 watts
	$=$	9.23 hp.

$$\text{Efficiency} = \frac{51.7 \times 100}{51.7 + 9.23} = \frac{51.7 \times 100}{60.93} = 84.9 \text{ per cent}$$

$$\text{Power factor} = \frac{60.93 \times 746 \times 100}{2{,}200 \times 21.12} = 97.7 \text{ per cent}$$

2. *More Exact Solution, Taking into Account No-load Current:*

$$Z_1 + Z^- = 2r_1 + \frac{r_2'}{2-s} + j(2x_1 + x_2)$$

$$= 2 \times 3.468 + \frac{3.316}{1.98} + j3 \times \frac{13.18}{2}$$

$$= 8.61 + j19.77$$

$$V_1' = V_1 - I_e(Z_1 + Z^-) \text{ vectorially}$$

$$\cong V_1 - I_e(2x_1 + x_2) \text{ algebraically}^1$$

$$= 2{,}200 - 7.36 \times 19.77 = 2{,}054.5 \text{ volts}$$

$$I_2^{\pm} = \frac{V_1'}{\sqrt{3}(Z^+ + Z^-)} = \frac{2{,}054.5}{\sqrt{3} \times 104.1} = 11.39 \text{ amp.}$$

[1] Since, for this particular motor, the resistance part of the impedance $Z_1 + Z^-$ is exceptionally high, being 43.6 per cent of the reactance, it might perhaps be doubted that algebraic subtraction of the no-load reactance

It appears, from the approximate solution, that the reverse torque is less than 2 per cent of the direct torque. Great accuracy in the negative-sequence current is, therefore, of little importance as far as torque and output are concerned. It affects, however, the efficiency and the power factor.

$$I_{\bar{1}} = \frac{I_1}{\sqrt{3}} = \frac{I_2 + I_e}{\sqrt{3}} = \frac{\sqrt{(\sqrt{3}\,I_2^+)^2 + (I_e)^2}}{\sqrt{3}} \text{ (approximately)}$$

$$= \sqrt{11.39^2 + \left(\frac{7.36}{\sqrt{3}}\right)^2} = \sqrt{147.8} = 12.15 \text{ amp. (approximately)}$$

Total internal torque

$$T = \frac{3p}{4\pi f_1}\left[(I_2^+)^2\frac{r_2}{s} - (I_{\bar{1}})^2\frac{r_2'}{2-s}\right]$$

$$= \frac{3 \times 6 \times 550}{4\pi \times 60 \times 746}\left[\frac{11.39^2 \times 1.842}{0.02} - \frac{12.15^2 \times 3.316}{1.98}\right]$$

$$= \frac{3 \times 6 \times 550}{4\pi \times 60 \times 746}[11{,}948 - 247.1] = 206.0 \text{ ft.-lb.}$$

$$\text{Pulley torque} = 206.0 - 5.4 = 200.6 \text{ ft.-lb.}$$

Power output

$$P = \frac{4\pi \times 60 \times 0.98 \times 200.6}{6 \times 550} = 44.8 \text{ hp.}$$

$$\text{Copper loss} = I_1^2 2r_1 + 3[(I_2^+)^2 r_2 + (I_{\bar{1}})^2 r_2']$$
$$= 3[(I_{\bar{1}})^2(2r_1 + r_2') + (I_2^+)^2 r_2]$$
$$= 3[12.15^2(2 \times 3.468 + 3.316) + 11.39^2 \times 1.842]$$
$$= 5{,}257 \text{ watts}$$

Core loss	= 600 watts
Friction and windage	= 900 watts
Total losses	= 6,757 watts
	= 9.05 hp.

---

drop from the terminal voltage will give sufficient accuracy. In order to show that the error introduced by this procedure is insignificant, the no-load impedance drop is subtracted vectorially below.

The no-load power factor is

$$\cos\theta_n = \frac{1{,}875}{2{,}200 \times 7.36} = 0.1158 \qquad \sin\theta_n = 0.9933$$

$$V_1' = 2{,}200 - 7.36(0.1158 - j0.9933)(8.61 + j19.77)$$
$$= 2{,}048.3 + j46.1 = 2{,}048.4 \text{ volts}$$

The discrepancy in V_1' is, hence,

$$\frac{2{,}054.5 - 2{,}048.4}{2{,}048.4}100 = 0.30 \text{ per cent}$$

which is negligible.

Subtracting the impedance drop algebraically gives

$$V_1' = 2{,}200 - 7.36\sqrt{8.61^2 + 19.77^2}$$
$$= 2{,}200 - 7.36 \times 21.55 = 2{,}041 \text{ volts}$$

The discrepancy in this case is

$$\frac{2{,}048.4 - 2{,}041.4}{2{,}048.4}100 = 0.34 \text{ per cent}$$

This discrepancy is also negligibly small. It seems, therefore, that it makes very little difference which method is used for the calculation of V_1'.

$$\text{Efficiency} = \frac{44.8 \times 100}{44.8 + 9.05} = \frac{44.8 \times 100}{53.85} = 83.2 \text{ per cent}$$

$$\text{Power factor} = \frac{53.85 \times 746 \times 100}{2{,}200 \times \sqrt{3} \times 12.15} = 86.9 \text{ per cent}$$

It is interesting to compare the results obtained by the two solutions, *viz.*, when excitation is neglected or taken into account. In order to facilitate comparison, the calculated results are gathered in the following table:

Items calculated	Solution 1, exciting current neglected	Solution 2, exciting current considered
Line current	21.12 amp	21.05 amp.
Pulley torque	231.1 ft.-lb.	200.6 ft.-lb.
Power output	51.7 hp.	44.8 hp.
Efficiency	84.9 per cent	83.2 per cent
Power factor	97.7 per cent	86.9 per cent

Approximate Determination of Performance from Circle Diagram.—Figure 91 shows the well-known circle diagram[1] of the three-phase induction motor. It is assumed that the reader is familiar with its construction and use. The diagram is intended to give the performance for operation on a balanced circuit and is constructed for some specific value of impressed voltage, usually the rated voltage of the motor. If the voltage is another, the scales will no longer be correct but can readily be adjusted so as to correspond to the new value of voltage. The current readings should be multiplied by the ratio of the actual voltage to the voltage on which the diagram is based, and the power and torque readings by the square of this ratio.

The performance of the motor on unbalanced voltages may be approximately determined from the diagram by reading off separately the power and torques due to the positive- and negative-sequence systems. It is assumed that the component voltages impressed are known so that the readings may be properly corrected.

The positive-sequence operating point is located by means of the actual slip of the machine at, say, point e in the diagram. The negative-sequence operating point is located where the slip is $2 - s$. This would be at point j, for instance. Having located the operating points the current, power, torque, and power factor for each system may be read. Since the negative-sequence

[1] See, for instance, "Principles of Alternating-current Machinery," by R. R. Lawrence, *loc. cit.*

excitation is negligible, the current for this sequence should be taken as proportional to the line aj rather than to the line oj from the origin of the diagram. Knowing current, power, etc., for the separate systems, they may be combined to give the actual unbalanced currents, the resultant power and torque and the efficiency.

When the circle diagram is available, this method of predicting performance on an unbalanced circuit is very simple, rapid, and

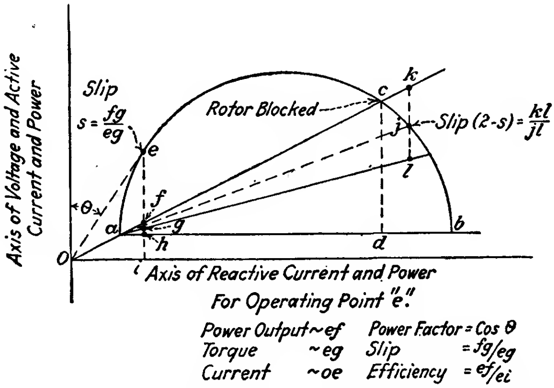

FIG. 91.—Circle diagram of a polyphase induction motor.

may give a precision which is sufficient in many instances. It should be borne in mind, however, that the circle diagram in general does not take into account the variation in rotor resistance with slip. It is constructed on the basis of a blocked test, during which the rotor resistance corresponds to impressed frequency. Hence, the diagram fails to consider the fact that, in the operating region, the rotor resistance is very nearly ohmic for the positive-sequence system and should correspond very nearly to double impressed frequency for the negative-sequence system.

CHAPTER VI

THE SHORT TRANSMISSION LINE IN THE STEADY STATE

Introduction.—A *smooth* line possesses four distinct electrical parameters or constants: *resistance, inductance, leakance*, and *capacitance*, all uniformly distributed along the line. A rigorous analysis of performance should obviously take all four parameters into account.

The leakance of aerial lines for power transmission is usually negligibly small; neither is the capacitance very large per unit length of such lines. The leakance depends on the power lost over the insulators and the corona loss. The former is quite insignificant and need seldom be considered. The latter will only come into effect in very high-voltage lines, and even there only when the operating voltage exceeds the disruptive critical voltage.[1] The capacitance depends only on the configuration of the line.

The effect of the capacitance on the performance of a *very short line* (say, below 40 miles in length), or *feeder*, is extremely small. Such lines, therefore, are treated as if they contained resistance and inductance only. In other words, they are considered equivalent to lumped impedances.

As the line becomes longer, the effect of the capacitance becomes more and more noticeable and should be accounted for, at least in an approximate way. Such lines of moderate length (about 40 to 100 miles in length) are often considered as having the capacitance lumped at certain points instead of uniformly distributed. The total capacitance may be considered lumped at the center of the line, in which case the *nominal T-line* results. Or one-half of the total capacitance may be lumped at each end. This gives the *nominal Π-line*. The capacitance may also be lumped at several other points along the line, the idea being to approach the actual smooth line more closely. In each case, the sum of the separate lumped capacitances is taken equal to the

[1] See, for instance, F. W. PEEK, JR., "Dielectric Phenomena in High-voltage Engineering," McGraw-Hill Book Company, Inc., New York, 1920.

total capacitance. For instance, two-thirds of the capacitance may be lumped at the center and one-sixth at each end. It may be said, however, that subdivision beyond this and maybe even beyond two lumped capacitances, as in the Π-line, is hardly worth while. If greater accuracy is required it is both easier and better to apply the exact long-line solutions presented in the next chapter.

In a cable, the leakance and capacitance per unit length, particularly the latter, are considerably higher than for an aerial line. Hence, only with very short cables (say, below 2 miles in length) can capacitance be neglected. For lengths between 2 miles and 5 miles (approximately), the calculations of performance may be based on the nominal T- or Π-circuits. For cables longer than specified above, exact methods of solution should be employed.

In the following are presented analytical solutions for the nominal T, the nominal Π, and for the case where two-thirds of the capacitance is lumped at the center and one-sixth at each end of the line.

Capacitance Lumped at Center (Nominal T).—The circuit is shown in Fig. 92. The impedance $Z/2$ of each arm of the T is

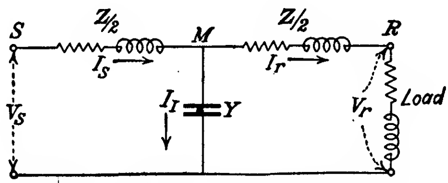

FIG. 92.—Nominal T representation of a short transmission line.

equal to one-half the impedance of the line. The admittance Y of the leak is equal to the total admittance of the line. This admittance would, as a rule, contain capacitance only.

In terms of the voltage and current at the load, the voltage and current at the sending end are given by

$$V_s = \left(1 + \frac{ZY}{2}\right)V_r + Z\left(1 + \frac{ZY}{4}\right)I_r \tag{1}$$

$$I_s = YV_r + \left(1 + \frac{ZY}{2}\right)I_r \tag{2}$$

Figure 93 gives the vector diagram for this circuit.

By means of equations (1) and (2) any problem involving power transmitted, efficiency, regulation, etc., can be solved.

When the load is removed, the voltage at the receiver end rises to

$$V_r' = \frac{V_s}{1 + \frac{ZY}{2}} \tag{3}$$

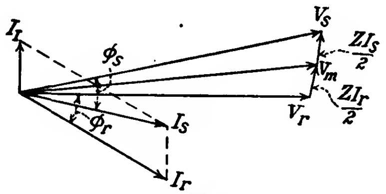

Fig. 93.—Vector diagram of voltages and currents in the T-circuit in Fig. 92.

The per cent voltage regulation is, hence,

$$\text{Regulation} = \frac{V_r' - V_r}{V_r} 100 \tag{4}$$

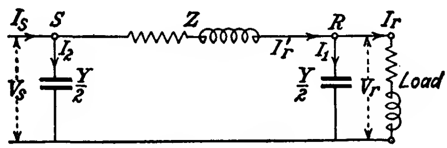

Fig. 94.—Nominal π representation of a short transmission line.

Capacitance Lumped at Ends (Nominal Π).—The nominal Π-circuit is shown in Fig. 94. Here the architrave impedance Z is equal to the total impedance, while each of the end leaks $Y/2$ is equal to one-half of the total admittance of the line which the Π-circuit replaces.

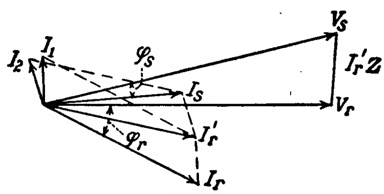

Fig. 95.—Vector diagram of voltages and currents in the Π-circuit in Fig. 94.

The sending voltage and current may be written

$$V_s = \left(1 + \frac{ZY}{2}\right)V_r + ZI_r \tag{5}$$

$$I_s = Y\left(1 + \frac{ZY}{4}\right)V_r + \left(1 + \frac{ZY}{2}\right)I_r \tag{6}$$

The vector diagram for the Π-circuit is given in Fig. 95.

The receiver voltage at no load and the voltage regulation are given by the same expressions as for the T-circuit (Equations (3) and (4)).

Capacitance Lumped at Center and at Ends.—Figure 96 illustrates a circuit in which two-thirds of the line admittance ($\frac{2}{3}Y$) is lumped at the midpoint and one-sixth ($Y/6$) at each end. The

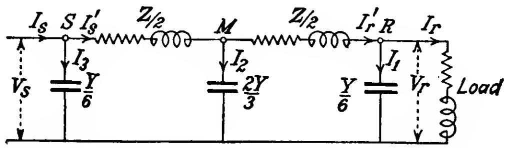

FIG. 96.—Improved representation of a short transmission line by lumping capacitance at center and at ends.

impedance of the architraves on each side of the midpoint admittance is one-half the impedance of the line ($Z/2$).

The voltage-current equations for this circuit become

$$V_s = \left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{36}\right)V_r + Z\left(1 + \frac{ZY}{6}\right)I_r \tag{7}$$

$$I_s = Y\left(1 + \frac{5ZY}{36} + \frac{Z^2Y^2}{216}\right)V_r + \left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{36}\right)I_r \tag{8}$$

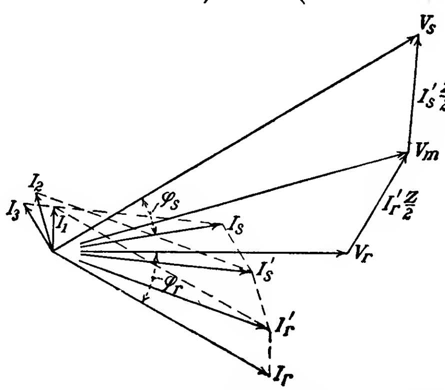

FIG. 97.—Vector diagram of voltages and currents in the circuit in Fig. 96.

The vector diagram is shown in Fig. 97.

When the load is removed, the receiver voltage rises to

$$V_r' = \frac{V_s}{1 + \dfrac{ZY}{6} + \dfrac{Z^2Y^2}{36}} \tag{9}$$

Substituting this value of receiver voltage in equation (4), the regulation is obtained.

EXAMPLE 1

Statement of Problem

A three-phase, 60-cycle transmission line, 100 miles long with 7-ft. spacing between conductors of No. 0 wire, has the following constants per wire mile:

$$r = 0.52 \text{ ohm}$$
$$x = 0.78 \text{ ohm}$$
$$y = j5.5 \times 10^{-6} \text{ mhos}$$

The power received is 4,500 kv.-a. at 80 per cent power factor (lagging) and 66,000 volts between lines. Calculate

a. The sending voltage.
b. The sending current.
c. The sending power.
d. The sending power factor.
e. The efficiency of transmission.
f. The per cent regulation.
g. The no-load charging current at 66,000 volts receiver voltage.
h. The no-load charging current at 66,000 volts sending voltage, using the following circuit arrangements:

1. The admittance lumped at the center.
2. One-half of the admittance lumped at each end.
3. Two-thirds of the admittance lumped at the center and one-sixth at each end.
4. Admittance uniformly distributed, *i.e.*, exact solution by smooth-line theory.

Solution

Total impedance and admittance per wire

$$Z = 52 + j78 \text{ ohms}$$
$$Y = j5.5 \times 10^{-4} \text{ mhos}$$

Total receiver power

$$P_R = 4{,}500 \times 0.8 = 3{,}600 \text{ kw.}$$

Receiver current

$$I_r = I_R = \frac{4{,}500}{\sqrt{3} \times 66}(0.8 - j0.6) = 31.49 - j23.62 \text{ amp. (per wire)}$$

1. *The Admittance Lumped at the Center.*

a. Sending Voltage:

$$V_S = \left(1 + \frac{ZY}{2}\right)V_R + \sqrt{3}Z\left(1 + \frac{ZY}{4}\right)I_R \qquad (a)$$

$$1 + \frac{ZY}{2} = 1 + j5.5 \times 10^{-4}(26 + j39) = 0.97855 + j0.0143$$
$$= 0.97865$$

$$Z\left(1 + \frac{ZY}{4}\right) = (52 + j78)\left(1 + \frac{j5.5 \times 10^{-4}(52 + j78}{4}\right)$$
$$= 50.886 + j77.635$$

$$V_S = (0.97855 + j0.0143)66{,}000 + \sqrt{3}(50.886 + j77.635)\ (31.49 - j23.62)$$
$$= \sqrt{3}(40.722 + j1{,}784.6) = 70{,}600 \text{ volts (between lines)}$$

b. Sending Current:

$$I_S = \frac{YV_R}{\sqrt{3}} + \left(1 + \frac{ZY}{2}\right)I_R \tag{b}$$

$$I_S = \frac{j5.5 \times 10^{-4} \times 66{,}000}{\sqrt{3}} + (0.97855 + j0.0143)(31.49 - j23.62)$$

$$= 31.152 - j1.705 = 31.20 \text{ amp.}$$

c. Total Sending Power:

$$P_S = 3(40{,}722 \times 31.152 - 1{,}784.6 \times 1.705)10^{-3} = 3{,}796.7 \text{ kw.}$$

d. Sending Power Factor:

$$\cos\theta_S = \frac{P_S}{\sqrt{3}V_SI_S} = \frac{3{,}796.7 \times 100}{\sqrt{3} \times 70{,}600 \times 31.20} = 99.52 \text{ per cent (lagging)}$$

e. Efficiency of Transmission:

$$\eta = \frac{P_R}{P_S} = \frac{3{,}600 \times 100}{3{,}796.7} = 94.82 \text{ per cent}$$

f. Per cent Voltage Regulation:

$$V_R' = \frac{V_S}{1 + \dfrac{ZY}{2}} = \frac{70{,}600}{0.97865} = 72{,}140 \text{ volts}$$

$$\text{Regulation} = \frac{V_R' - V_R}{V_R} = \frac{72{,}140 - 66{,}000}{66{,}000}100 = 9.30 \text{ per cent}$$

g. No-load Charging Current at 66,000 Volts Receiver Voltage:

$$I_S = \frac{YV_R}{\sqrt{3}} = \frac{j5.5 \times 10^{-4} \times 66{,}000}{\sqrt{3}} = j20.958 = 20.96 \text{ amp.}$$

h. No-load Charging Current at 66,000 Volts Sending Voltage:

$$I_S = \frac{YV_S}{\sqrt{3}\left(1 + \dfrac{ZY}{2}\right)} = \frac{20.96}{0.97865} = 21.40 \text{ amp.}$$

2. *One-half of the Admittance Lumped at Each End.*

a. Sending Voltage:

$$V_S = \left(1 + \frac{ZY}{2}\right)V_R + \sqrt{3}ZI_R \tag{c}$$

$$V_S = (0.97855 + j0.0143)66{,}000 + \sqrt{3}(52 + j78)(31.49 - j23.62)$$

$$= \sqrt{3}(40{,}768 + j1{,}772.9) = 70{,}679 \text{ volts (between lines)}$$

b. Sending Current:

$$I_S = Y\left(1 + \frac{ZY}{4}\right)\frac{V_R}{\sqrt{3}} + \left(1 + \frac{ZY}{2}\right)I_R \tag{d}$$

$$Y\left(1 + \frac{ZY}{4}\right) = j5.5 \times 10^{-4}\left(1 + \frac{j5.5 \times 10^{-4}(52 + j78)}{4}\right)$$

$$= (-0.0393 + j5.4410)10^{-4}$$

$$I_S = (-0.0393 + j5.4410)10^{-4} \times \frac{66{,}000}{\sqrt{3}} + (0.97855 + j0.0143)(31.44 - j23.62)$$

$$= 31.003 - j1.930 = 31.06 \text{ amp.}$$

c. Total Sending Power:

$$P_S = 3(40{,}768 \times 31.003 - 1{,}772.9 \times 1.930)10^{-3} = 3{,}781.5 \text{ kw.}$$

d. Sending Power Factor:

$$\cos\theta_S = \frac{P_S}{\sqrt{3}V_SI_S} = \frac{3{,}781.5 \times 100}{\sqrt{3} \times 70.679 \times 31.06} = 99.45 \text{ per cent (lagging)}$$

e. Efficiency of Transmission:

$$\eta = \frac{P_R}{P_S} = \frac{3{,}600 \times 100}{3{,}781.5} = 95.20 \text{ per cent}$$

f. Per Cent Voltage Regulation:

$$V_R' = \frac{V_S}{1 + \dfrac{ZY}{2}} = \frac{70{,}679}{0.97865} = 72{,}200 \text{ volts}$$

$$\text{Regulation} = \frac{V_R' - V_R}{V_R} = \frac{72{,}200 - 66{,}000}{66{,}000} 100 = 9.43 \text{ per cent}$$

g. No-load Charging Current at 66,000 Volts Receiver Voltage:

$$I_S = Y\left(1 + \frac{ZY}{4}\right)\frac{V_R}{\sqrt{3}} = (-0.0393 + j5.4410)10^{-4} \times \frac{66{,}000}{\sqrt{3}}$$
$$= -0.1498 + j20.733 = 20.74 \text{ amp.}$$

h. No-load Charging Current at 66,000 Volts Sending Voltage:

$$I_S = \frac{Y\left(1 + \dfrac{ZY}{4}\right)V_S}{\sqrt{3}\left(1 + \dfrac{ZY}{2}\right)} = \frac{20.74}{0.97865} = 21.19 \text{ amp.}$$

3. *Two-thirds of the Admittance Lumped at the Center and One-sixth at Each End.*

a. Sending Voltage:

$$V_S = \left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{36}\right)V_R + \sqrt{3}Z\left(1 + \frac{ZY}{6}\right)I_R \tag{e}$$

$$ZY = j5.5 \times 10^{-4}(52 + j78) = (-429 + j286)10^{-4}$$
$$Z^2Y^2 = (102{,}245 - j245{,}388)10^{-8}$$
$$1 + \frac{ZY}{2} + \frac{Z^2Y^2}{36} = 1 + (-214.5 + j143)10^{-4} + (2{,}840 - j6{,}816)10^{-8}$$
$$= 0.97858 + j0.0142 = 0.9787$$
$$Z\left(1 + \frac{ZY}{6}\right) = (52 + j78)(1 - 0.00715 + j0.00477) = 51.256 + j77.69$$

$$V_S = (0.97858 + j0.0142)66{,}000 + \sqrt{3}(51.256 + j77.69)(31.49 - j23.62)$$
$$= 70{,}560 + j3{,}078 = 70{,}627 \text{ volts (between lines)}$$

b. Sending Current:

$$I_S = Y\left(1 + \frac{5ZY}{36} + \frac{Z^2Y^2}{216}\right)\frac{V_R}{\sqrt{3}} + \left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{36}\right)I_R \tag{f}$$

$$Y\left(1 + \frac{5ZY}{36} + \frac{Z^2Y^2}{216}\right) = j5.5 \times 10^{-4}(1 - 0.00596 + j0.00397 + 0.00000079 - j0.0000019)$$
$$= (-0.02183 + j5.4672)10^{-4}$$

$$I_S = (-0.02183 + j5.4672)10^{-4} \times \frac{66{,}000}{\sqrt{3}} + (0.97858 + j0.0142)(31.49 - j23.62)$$
$$= \frac{1}{\sqrt{3}}(53.814 - j3.174) = 31.12 \text{ amp.}$$

c. Total Sending Power:

$$P_S = (70{,}560 \times 53.814 - 3{,}078 \times 3.174)10^{-3} = 3{,}787.4 \text{ kw.}$$

d. Sending Power Factor:

$$\cos \theta_S = \frac{P_S}{\sqrt{3} V_S I_S} = \frac{3{,}787.4 \times 100}{\sqrt{3} \times 70.627 \times 31.12} = 99.47 \text{ per cent (lagging)}$$

e. Efficiency of Transmission:

$$\eta = \frac{P_R}{P_S} = \frac{3{,}600 \times 100}{3{,}787.4} = 95.05 \text{ per cent}$$

f. Per Cent Voltage Regulation:

$$V_R' = \frac{V_S}{1 + \frac{ZY}{2} + \frac{Z^2Y^2}{36}} = \frac{70{,}627}{0.9787} = 72{,}165 \text{ volts}$$

$$\text{Regulation} = \frac{V_R' - V_R}{V_R} = \frac{72{,}165 - 66{,}000}{66{,}000} 100 = 9.34 \text{ per cent}$$

g. No-load Charging Current at 66,000 Volts Receiver Voltage:

$$I_S = Y\left(1 + \frac{5ZY}{36} + \frac{Z^2Y^2}{216}\right)\frac{V_R}{\sqrt{3}} = (-0.02183 + j5.4672)10^{-4} \times \frac{66{,}000}{\sqrt{3}}$$

$$= -0.0832 + j20.833 = 20.83 \text{ amp.}$$

h. No-load Charging Current at 66,000 Volts Sending Voltage:

$$I_S = \frac{Y\left(1 + \frac{5ZY}{36} + \frac{Z^2Y^2}{216}\right)V_S}{\sqrt{3}\left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{36}\right)} = \frac{20.83}{0.9787} = 21.28 \text{ amp.}$$

4. *Solution by Exact Formulas.*

a. Sending Voltage:

$$V_S = \cosh \theta \, V_R + Z_0 \sinh \theta \, I_R \qquad (g)$$

$$\theta = \sqrt{ZY} = \sqrt{(52 + j78)j5.5 \times 10^{-4}} = 0.227\underline{/73°.15}$$

$$= 0.0658 + j0.2173 \text{ hyp.}$$

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{52 + j78}{j5.5 \times 10^{-4}}} = 412.7\overline{\backslash 16°.85} \text{ ohms}$$

Calculating the hyperbolic functions by equations (87) and (88) in Chap. VII gives

$$\cosh \theta = 1.0022 \times 0.9765 + j0.0658 \times 0.2155$$

$$= 0.97865 + j0.0141 = 0.97875$$

$$\sinh \theta = 0.0658 \times 0.9765 + j1.0022 \times 0.2155$$

$$= 0.0643 + j0.216 = 0.2254\underline{/73°.45}$$

$$V_S = (0.97865 + j0.0141)66{,}000 +$$

$$\sqrt{3} \times 412.7\overline{\backslash 16°.85} \times 0.2254\underline{/73°.45}(31.49 - j23.62)$$

$$= 70{,}562 + j3{,}082.6 = 70{,}629 \text{ volts (between lines)}$$

b. Sending Current:

$$I_S = \frac{\sinh \theta}{Z_0}\frac{V_R}{\sqrt{3}} + \cosh \theta \, I_R \qquad (h)$$

$$I_S = \frac{0.2254\underline{/73°.45} \times 66{,}000}{\sqrt{3} \times 412.7\overline{\backslash 16°.85}} + (0.97865 + j0.0141)(31.49 - j23.62)$$

$$= \frac{1}{\sqrt{3}}(53.771 - j3.22) = 31.10 \text{ amp.}$$

c. Total Sending Power:

$$P_S = (70{,}562 \times 53.771 - 3{,}082.6 \times 3.22)10^{-3} = 3{,}784.3 \text{ kw.}$$

d. Sending Power Factor:

$$\cos \theta_S = \frac{P_S}{\sqrt{3}V_S I_S} = \frac{3{,}784.3 \times 100}{\sqrt{3} \times 70.629 \times 31.10} = 99.47 \text{ per cent (lagging)}$$

e. Efficiency of Transmission:

$$\eta = \frac{P_R}{P_S} = \frac{3{,}600 \times 100}{3{,}784.3} = 95.13 \text{ per cent}$$

f. Per cent Voltage Regulation:

$$V_R' = \frac{V_S}{\cosh \theta} = \frac{70{,}629}{0.97875} = 72{,}163 \text{ volts}$$

$$\text{Regulation} = \frac{V_R' - V_R}{V_R} = \frac{72{,}163 - 66{,}000}{66{,}000}100 = 9.37 \text{ per cent}$$

g. No-load Charging Current at 66,000 Volts Receiver Voltage:

$$I_S = \frac{\sinh \theta}{Z_0}\frac{V_R}{\sqrt{3}} = \frac{0.2254\underline{/73°.45} \times 66{,}000}{\sqrt{3} \times 412.7\overline{\backslash 16°.85}}$$

$$= \frac{36.047}{\sqrt{3}}\underline{/90°.3} = 20.81 \text{ amp.}$$

h. No-load Charging Current at 66,000 Volts Sending Voltage:

$$I_S = \frac{\sinh \theta \, V_S}{\sqrt{3}Z_0 \cosh \theta} = \frac{20.81}{0.97875} = 21.27 \text{ amp.}$$

Table VIII shows the results which have been calculated above using four different circuit arrangements. Comparison of the figures for the approximate lumped-constant circuits with the corresponding ones for the correct circuit with distributed constants shows that the agreement is very good. In the case where the capacitance is lumped at three points, almost exact results are obtained. When the nominal T-circuit or Π-circuit is used, the discrepancies are not large and the accuracy would undoubtedly be sufficient for most engineering purposes.

TABLE VIII

Item calculated	Circuit arrangement			
	Admittance lumped at center (nominal *T*)	One-half of admittance lumped at each end (nominal Π)	Two-thirds of admittance lumped at center, one-sixth at each end	Exact
a. Sending voltage.....	70,600 volts	70,679 volts	70,627 volts	70,629 volts
b. Sending current......	31.20 amp.	31.06 amp.	31.12 amp.	31.10 amp.
c. Sending power.......	3,796.7 kw.	3,781.5 kw.	3,787.4 kw.	3,784.3 kw.
d. Sending power factor.	99.52 per cent	99.45 per cent	99.47 per cent	99.47 per cent
e. Efficiency of transmission............	94.82 per cent	95.20 per cent	95.05 per cent	95.13 per cent
f. Per cent regulation...	9.30 per cent	9.43 per cent	9.34 per cent	9.37 per cent
g. No-load charging current at 66,000 volts receiver voltage......	20.96 amp.	20.74 amp.	20.83 amp.	20.81 amp.
h. No-load charging current at 66,000 volts sending voltage......	21.40 amp.	21.19 amp.	21.28 amp.	21.27 amp.

CHAPTER VII

THE LONG TRANSMISSION LINE IN THE STEADY STATE

A rigorous analysis of transmission-line performance should, as previously pointed out, recognize and take into account all four parameters of the line, *viz.*, resistance, inductance, leakance, and capacitance. If the line is short, the effect of leakance and capacitance may be neglected; if the line is long or, more correctly, *electrically long*, these two parameters should be considered.[1]

It should be noted that, while the physical length and the electrical length are identical for direct-current lines, they may differ greatly for alternating-current lines. Indeed, a very short alternating-current line may be electrically long. This is readily understood when it is remembered that the susceptance, which is by far the predominant part of the dielectric admittance, is directly proportional to the frequency. Hence, the electrical length of an alternating-current line depends on the frequency as well as on its actual physical length. Evidently the direct-current steady-state performance depends upon resistance and leakance alone; inductance and capacitance have no effect.

When the symbolic notation is employed a direct-current voltage (or current) may be considered equivalent to an alternating-current voltage (or current) of zero frequency. Hence, it is sufficient to work out the solutions for alternating currents; the formulas obtained are then directly applicable to direct currents by substituting zero for the frequency (or angular velocity) in the terms where it appears. A single set of formulas is thus capable of covering the alternating-current as well as the direct-current case.

[1] The following books treat the theory of long transmission lines in the steady state:

KENNELLY, A. E., "The Application of Hyperbolic Functions to Electrical-engineering Problems," McGraw-Hill Book Company, Inc., New York, 1912 and 1925.

KENNELLY, A. E., "Artificial Electric Lines," McGraw-Hill Book Company, Inc., New York, 1917.

PERNOT, F. E., "Electrical Phenomena in Parallel Conductors," John Wiley & Sons, Inc., New York, 1918.

General Solution.—Consider two parallel wires $S - R$ as shown in Fig. 98, having the following constants per loop mile:

$$\begin{aligned} \text{Resistance} &= r \text{ ohms} \\ \text{Inductance} &= L \text{ henries} \\ \text{Leakance} &= g \text{ mhos} \\ \text{Capacitance} &= C \text{ farads} \end{aligned}$$

At a point x miles from the receiver end R, the following differential equations may be established by considering the change in voltage and current in a differential element dx

$$\frac{\partial e}{\partial x} = ri + L\frac{\partial i}{\partial t} \tag{1}$$

$$\frac{\partial i}{\partial x} = ge + C\frac{\partial e}{\partial t} \tag{2}$$

These equations are perfectly general. They are written in terms of instantaneous voltage and current and are applicable to the transient, as well as to the steady state. In the alternating-

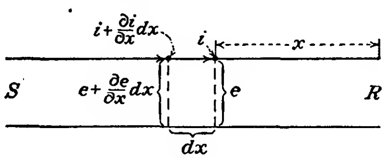

FIG. 98.—Section of a two-wire transmission line.

current, steady-state case, however, both voltage and current undergo sinusoidal variations with time. Equations (1) and (2) may, hence, be written

$$\frac{dE_x}{dx} = (r + j\omega L)I_x = zI_x \tag{3}$$

$$\frac{dI_x}{dx} = (g + j\omega C)E_x = yE_x \tag{4}$$

In these equations, E and I may represent either maximum or root-mean-square values of voltage and current.

The four differential equations just given were established for two parallel wires; in other words, for a single-phase line. Exactly the same equations, however, will be obtained for a three-phase line if the constants r, L, g, and C represent the equivalent three-phase constants, *i.e.*, the resistance per wire mile and the inductance, leakance, and capacitance per wire mile to neutral. The voltage and current are then to neutral and per wire, respectively. Hence, the formulas which are derived below are applicable to either case (and, as a matter of fact, also to other

polyphase lines) as long as the proper values are assigned to the constants.

By differentiation of equations (3) and (4) with respect to x is obtained

$$\frac{d^2E_x}{dx^2} = z\frac{dI_x}{dx} \tag{5}$$

$$\frac{d^2I_x}{dx^2} = y\frac{dE_x}{dx} \tag{6}$$

Substituting equation (4) in equation (5) and equation (3) in equation (6) gives

$$\frac{d^2E_x}{dx^2} = zyE_x = \alpha^2E_x \tag{7}$$

$$\frac{d^2I_x}{dx^2} = zyI_x = \alpha^2I_x \tag{8}$$

where

$$\alpha = \sqrt{zy} = \sqrt{(r + j\omega L)(g + j\omega C)} \tag{9}$$

The quantity α is termed the *attentuation constant* of the line.

Equations (7) and (8) are recognized as simple linear differential equations of the second order with constant coefficients. Their solution may be obtained either in terms of exponential or hyperbolic functions.[1] The latter is preferable in line calculations and may be written

$$E_x = A \cosh \alpha x + B \sinh \alpha x \tag{10}$$

$$I_x = C \cosh \alpha x + D \sinh \alpha x \tag{11}$$

The constants of integration are evaluated by making use of the terminal conditions.

When $x = 0$, $E_x = E_r$ and $I_x = I_r$

Hence A and C are immediately obtained as

$$A = E_r \quad \text{and} \quad C = I_r$$

In order to determine B and D, equations (10) and (11) are differentiated with respect to x

$$\frac{dE_x}{dx} = A\alpha \sinh \alpha x + B\,\alpha \cosh \alpha x \tag{12}$$

$$\frac{dI_x}{dx} = C\alpha \sinh \alpha x + D\,\alpha \cosh \alpha x \tag{13}$$

Substituting equations (3) and (4) in equations (12) and (13) gives, when $x = 0$

$$B\alpha = zI_r \tag{14}$$

$$D\alpha = yE_r \tag{15}$$

[1] See any standard treatise on differential equations.

from which

$$B = \frac{z}{\alpha} I_r = Z_0 I_r$$

$$D = \frac{y}{\alpha} E_r = \frac{E_r}{Z_0}$$

where

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{r + j\omega L}{g + j\omega C}} \tag{16}$$

Z_0 has the dimensions of an impedance and is called the *surge impedance* of the line.

Inserting the constants in equations (10) and (11) gives

$$E_x = E_r \cosh \alpha x + Z_0 I_r \sinh \alpha x \tag{17}$$

$$I_x = I_r \cosh \alpha x + \frac{E_r}{Z_0} \sinh \alpha x \tag{18}$$

If the conditions at the sending end had been used in determining the constants of integration, the following equations would have been obtained:

$$E_x = E_s \cosh \alpha(l - x) - Z_0 I_s \sinh \alpha(l - x) \tag{19}$$

$$I_x = I_s \cosh \alpha(l - x) - \frac{E_s}{Z_0} \sinh \alpha(l - x) \tag{20}$$

Equations (17) to (20) give the voltage and current distribution along the line with reference to voltage and current at the receiver

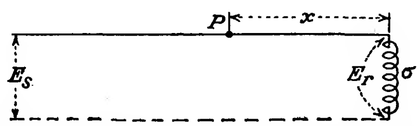

FIG. 99.—Single-phase representation of a long transmission line loaded with an impedance σ.

end and sending end respectively. They are perfectly general for sinusoidal, steady-state conditions and will hold whether the line is loaded, free, or grounded at the receiver end.

Although these equations as they stand are convenient for many purposes, they may be reduced to more compact forms. These shorter forms, which are still more suitable for computations, are somewhat different for the various conditions that may obtain at the receiver terminals.

Line Loaded.—Let the load at the receiver end be represented by its equivalent impedance σ as indicated in Fig. 99. The receiver-end current is then the receiver-end voltage E_r divided

by this impedance, and by making the proper substitution in equations (17) and (18) these may be written

$$E_x = E_r\left[\cosh \alpha x + \frac{Z_0}{\sigma} \sinh \alpha x\right] \tag{21}$$

$$I_x = I_r\left[\cosh \alpha x + \frac{\sigma}{Z_0} \sinh \alpha x\right] \tag{22}$$

Introducing a quantity θ', the so-called *hyperbolic angle of the load* defined by

$$\theta' = \tanh^{-1} \frac{\sigma}{Z_0} \tag{23}$$

in equations (21) and (22) gives

$$\begin{aligned} E_x &= E_r\,[\cosh \alpha x + \coth \theta' \sinh \alpha x] \\ &= E_r \frac{\cosh \alpha x \sinh \theta' + \sinh \alpha x \cosh \theta'}{\sinh \theta'} \\ &= E_r \frac{\sinh (\alpha x + \theta')}{\sinh \theta'} = E_r \frac{\sinh \delta_x}{\sinh \theta'} \end{aligned} \tag{24}$$

$$\begin{aligned} I_x &= I_r\,[\cosh \alpha x + \tanh \theta' \sinh \alpha x] \\ &= I_r \frac{\cosh \alpha x \cosh \theta' + \sinh \alpha x \sinh \theta'}{\cosh \theta'} \\ &= I_r \frac{\cosh (\alpha x + \theta')}{\cosh \theta'} = I_r \frac{\cosh \delta_x}{\cosh \theta'} \end{aligned} \tag{25}$$

The δ's are termed *position angles* and represent the total hyperbolic angle at the point considered (distance x from receiver end), being equal to the sum of the hyperbolic angle θ' of the load and the hyperbolic angle αx of the portion of the line between the point and the load.

The relation between the voltages and currents at the sending and receiving ends is evidently obtained from equations (24) and (25) by placing $x = l$. Hence,

$$E_s = E_r \frac{\sinh (\alpha l + \theta')}{\sinh \theta'} = E_r \frac{\sinh \delta_s}{\sinh \theta'} \tag{26}$$

$$I_s = I_r \frac{\cosh (\alpha l + \theta')}{\cosh \theta'} = I_r \frac{\cosh \delta_s}{\cosh \theta'} \tag{27}$$

By substituting the values of receiving-end voltage and current from equations (26) and (27) in equations (24) and (25), the voltage and current at any point are obtained in terms of sending voltage and current.

$$E_x = E_s \frac{\sinh \delta_x}{\sinh \delta_s} \tag{28}$$

$$I_x = I_s \frac{\cosh \delta_x}{\cosh \delta_s} \tag{29}$$

In a similar manner, the voltage and current at any point may be written in terms of the voltage and current at any arbitrary point. It is evident that the following general rule will hold: *The ratio of the voltages at two points of a line is equal to the ratio of the hyperbolic sines of the position angles at the two points and the ratio of currents is equal to the ratio of the hyperbolic cosines of the same position angles.*

The variation in vector power (volt-amperes) along the line is readily obtained by making use of either equations (24) and (25) or equations (28) and (29)

$$\text{Vector } P_x = E_x I_x = E_r I_r \frac{\sinh \delta_x \cosh \delta_x}{\sinh \theta' \cosh \theta'}$$

$$= E_r I_r \frac{\sinh 2\delta_x}{\sinh 2\theta'} = E_s I_s \frac{\sinh 2\delta_x}{\sinh 2\delta_s} \tag{30}$$

As seen, the vector power at a point is proportional to the hyperbolic sine of twice the position angle. The active power P and reactive power Q at any point are most readily computed by multiplying the product of the numerical values of voltage and current at the point by the cosine and sine of the angle between them. These quantities may also be obtained by multiplying the vector current I_x by the conjugate of the vector voltage E_x.

The impedance at any point offered by the line (including the load) is given by

$$Z_x = \frac{E_x}{I_x} = \frac{E_r}{I_r} \frac{\dfrac{\sinh \delta_x}{\sinh \theta'}}{\dfrac{\cosh \delta_x}{\cosh \theta'}} = \sigma \coth \theta' \tanh \delta_x = Z_0 \tanh \delta_x \tag{31}$$

and, similarly, the sending-end impedance becomes

$$Z_s = Z_0 \tanh \delta_s \tag{32}$$

Line Grounded.—A grounded line is equivalent to a loaded line where the impedance of the load is zero. The position angle of the load, therefore, is

$$\theta' = \tanh^{-1} \frac{0}{Z_0} = 0$$

Hence, from equations (28) and (29),

$$E_x = E_s \frac{\sinh \delta_x}{\sinh \delta_s} = E_s \frac{\sinh \theta_x}{\sinh \theta} \tag{33}$$

$$I_x = I_s \frac{\cosh \delta_x}{\cosh \delta_s} = I_s \frac{\cosh \theta_x}{\cosh \theta} \tag{34}$$

where

$$\theta_x = \alpha x \text{ and } \theta = \alpha l \tag{35}$$

The sending-end impedance is for the grounded line

$$Z_s = Z_0 \tanh \theta \tag{36}$$

The receiver voltage is evidently zero. The receiver current is given by

$$I_r = \frac{I_s}{\cosh \theta} = \frac{E_s}{Z_0 \sinh \theta} \tag{37}$$

The current at any point in terms of the receiver current becomes

$$I_x = I_r \cosh \theta_x \tag{38}$$

Line Free.—A line open at the distant end is equivalent to a loaded line where the impedance of the load is infinite. The position angle of such a load is

$$\theta' = \tanh^{-1} \infty = j\frac{\pi}{2}$$

Using equations (28) and (29), the voltage and current at some point on the line are obtained in terms of the voltage and current at the sending end

$$E_x = E_s \frac{\sinh \delta_x}{\sinh \delta_s} = E_s \frac{\sinh\left(\theta_x + j\frac{\pi}{2}\right)}{\sinh\left(\theta + j\frac{\pi}{2}\right)} = E_s \frac{\cosh \theta_x}{\cosh \theta} \tag{39}$$

$$I_x = I_s \frac{\cosh \delta_x}{\cosh \delta_s} = I_s \frac{\cosh\left(\theta_x + j\frac{\pi}{2}\right)}{\cosh\left(\theta + j\frac{\pi}{2}\right)} = I_s \frac{\sinh \theta_x}{\sinh \theta} \tag{40}$$

The sending-end impedance is for the open line

$$Z_s = Z_0 \tanh\left(\theta + j\frac{\pi}{2}\right) = Z_0 \coth \theta \tag{41}$$

The receiver-end current is zero. The receiver voltage is given by

$$E_r = \frac{E_s}{\cosh \theta} \tag{42}$$

From this equation may be computed the *Ferranti effect* of the line, *i.e.*, the numerical ratio of receiving to sending voltage when the line is free. For lines of the lengths and frequencies used for commercial power transmission today, the numerical value of $\cosh \theta$ is always less than unity. Hence, at no load the receiver voltage is larger than the sending voltage. This difference may, for the longest lines in operation (about 300 miles), be quite appreciable.

The voltage at any point in terms of the receiver voltage becomes

$$E_x = E_r \cosh \theta_x \tag{43}$$

Load Impedance Equal to Surge Impedance. The Infinite Line.—When the impedance of the load is equal to the surge impedance of the line (both in magnitude and phase) the position angle of the load becomes infinite.

$$\theta' = \tanh^{-1} \frac{\sigma}{Z_0} = \tanh^{-1} = \infty$$

The voltage and current distribution may be obtained from equations (28) and (29), which may be written

$$E_x = E_s \frac{\sinh(\theta_x + \theta')}{\sinh(\theta + \theta')} = E_s \frac{\sinh \theta_x \cosh \theta' + \cosh \theta_x \sinh \theta'}{\sinh \theta \cosh \theta' + \cosh \theta \sinh \theta'} \tag{44}$$

$$I_x = I_s \frac{\cosh(\theta_x + \theta')}{\cosh(\theta + \theta')} = I_s \frac{\cosh \theta_x \cosh \theta' + \sinh \theta_x \sinh \theta'}{\cosh \theta \cosh \theta' + \sinh \theta \sinh \theta'} \tag{45}$$

Making use of the following relation, however:

$$\operatorname*{Limit}_{\theta' \to \infty} (\sinh \theta') = \operatorname*{Limit}_{\theta' \to \infty} (\cosh \theta')$$

these equations may be simplified by canceling out sinh θ' and cosh θ' in numerator and denominator. Hence,

$$E_x = E_s \frac{\sinh \theta_x + \cosh \theta_x}{\sinh \theta + \cosh \theta} \tag{46}$$

$$I_x = I_s \frac{\cosh \theta_x + \sinh \theta_x}{\cosh \theta + \sinh \theta} \tag{47}$$

But

$$\sinh \theta + \cosh \theta = \frac{\epsilon^\theta - \epsilon^{-\theta}}{2} + \frac{\epsilon^\theta + \epsilon^{-\theta}}{2} = \epsilon^\theta$$

Inserting this in equations (46) and (47) gives

$$E_x = E_u = E_s \frac{\epsilon^{\theta_x}}{\epsilon^\theta} = E_s \epsilon^{-(\theta - \theta_x)} = E_s \epsilon^{-\alpha(l-x)} = E_s \epsilon^{-\theta_u} \tag{48}$$

$$I_x = I_u = I_s \frac{\epsilon^{\theta_x}}{\epsilon^\theta} = I_s \epsilon^{-(\theta - \theta_x)} = I_s \epsilon^{-\alpha(l-x)} = I_s \epsilon^{-\theta_u} \tag{49}$$

θ_u is here the hyperbolic angle of the line from the *sending end* to the point in question. The equations show that the voltage and current distribution are exponential when the load impedance and surge impedance are equal. Such a line is said to have normal attenuation[1] and the exponential $\epsilon^{-\theta_u}$ is called the normal attenuation factor. Evidently, in the general case, the attenuation factor is given by the ratio of two hyperbolic functions.

[1] See Dr. Kennelly's "Artificial Electric Lines," *loc. cit.*

As a rule θ_u will be a complex quantity which may be written

$$\theta_u = \theta_{u1} + j\theta_{u2} \tag{50}$$

The exponential in equations (48) and (49) may therefore be broken up into two parts and the equations written

$$E_u = E_s\epsilon^{-\theta_{u1}}\epsilon^{-j\theta_{u2}} = E_s\epsilon^{-\theta_{u1}}\overline{\backslash\theta_{u2}} \tag{51}$$

$$I_u = I_s\epsilon^{-\theta_{u1}}\epsilon^{-j\theta_{u2}} = I_s\epsilon^{-\theta_{u1}}\overline{\backslash\theta_{u2}} \tag{52}$$

Here the real exponential $\epsilon^{-\theta_{u1}}$ gives the attenuation in magnitude while the angle θ_{u2} directly indicates the swing in phase of the voltage and current with respect to the corresponding quantities at the sending end.

The sending-end impedance becomes

$$Z_s = Z_0 \tanh(\theta + \infty) = Z_0 \tanh \infty = Z_0 \tag{53}$$

Also,

$$Z_x = Z_0 \tanh(\theta_x + \infty) = Z_0 \tanh \infty = Z_0 \tag{54}$$

Hence, the impedance offered at any arbitrary point on the line is always equal to the surge impedance.

It is interesting to note that the line loaded by an impedance equal to the surge impedance behaves, electrically, as an infinite line. On the latter the line angle at any point θ_x must obviously be infinite, independent of the position of the point. Considering the hypothetical possibility of a load on the infinite line, then the position angle of this load would always be finite except when the load impedance is equal to the surge impedance. In the latter case, the infinite line would itself be a "line loaded by an impedance equal to the surge impedance," and by virtue of this fact behave as such a line. For any other load impedance, however (including zero and infinity corresponding to the line being grounded and open respectively), the load-position angle θ' will be finite and may be ignored in equations (44) and (45) as compared to θ_x and θ. Hence, for the infinite line,

$$E_x = E_s\frac{\sinh \theta_x}{\sinh \theta} = E_s\frac{\epsilon^{\theta_x} - \epsilon^{-\theta_x}}{\epsilon^{\theta} - \epsilon^{-\theta}} \tag{55}$$

$$I_x = I_s\frac{\cosh \theta_x}{\cosh \theta} = I_s\frac{\epsilon^{\theta_x} + \epsilon^{-\theta_x}}{\epsilon^{\theta} + \epsilon^{-\theta}} \tag{56}$$

Here the exponentials with negative exponents are negligible compared to the ones having positive exponents, so, consequently,

$$E_u = E_s\frac{\epsilon^{\theta_x}}{\epsilon^{\theta}} = E_s\epsilon^{-(\theta-\theta_x)} = E_s\epsilon^{-\theta_u} \tag{57}$$

$$I_u = I_s\frac{\epsilon^{\theta_x}}{\epsilon^{\theta}} = I_s\epsilon^{-(\theta-\theta_x)} = I_s\epsilon^{-\theta_u} \tag{58}$$

These equations for the infinite line and equations (48) and (49) for the line loaded by an impedance equal to the surge impedance are, as seen, identical.

Evidently the impedance at any point of the infinite line, including the home end, is equal to the surge impedance. For instance,

$$Z_s = Z_0 \tanh (\theta_x + \theta') = Z_0 \tanh (\infty + \theta') = Z_0 \tanh \infty = Z_0 \tag{59}$$

Wave Length.—The conception of wave length is based on the line having normal attenuation, *i.e.*, on the infinite line or the line whose load impedance equals the surge impedance. The wave length λ is then the length in which the phase swing (or phase attenuation, as it is sometimes termed) amounts to 360 deg. or 2π radians.

$$\lambda\alpha_2 = \theta_{u2} = 2\pi$$

$$\lambda = \frac{2\pi}{\alpha_2} \tag{60}$$

A line of length λ is a one-wavelength line, of length $\lambda/2$ a half-wavelength line, and of length $\lambda/4$ a quarter-wavelength line.

These lines, which are all beyond the length used in power transmission today, have their peculiar and interesting characteristics. It should be noted, however, that, although quarter-wavelength effects, half-wavelength effects, etc., are not realizable in practice for the fundamental frequency, they may easily obtain for some of the harmonics. The space in this treatise, however, does not permit a detailed discussion of these features. An idea of the length necessary to produce quarter-wave effects may be had from the following example:

A certain 220-kv. transmission line has an attenuation constant at 60 cycles per second of

$$\alpha = \alpha_1 + j\alpha_2 = 0.000167 + j0.002075$$

The length necessary to make this line a quarter-wave line is consequently

$$l_{1/4} = \frac{\lambda}{4} = \frac{\pi}{2\alpha_2} = \frac{\pi}{2 \times 0.002075} = 757 \text{ miles} \tag{61}$$

A Few Remarks on the Hyperbolic Functions. Some Conversion Formulas.—The hyperbolic functions are most conveniently defined by their exponential equivalents, *viz.*,

$$\sinh x = \frac{\epsilon^x - \epsilon^{-x}}{2} \tag{62}$$

$$\cosh x = \frac{\epsilon^x + \epsilon^{-x}}{2} \tag{63}$$

By making use of the power series of the exponential functions

$$\epsilon^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{64}$$

$$\epsilon^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \tag{65}$$

the series representing the hyperbolic functions may easily be formed.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{66}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{67}$$

The corresponding well-known series of the trigonometric functions are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{68}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{69}$$

from which, in conjunction with the following series of the exponentials with imaginary exponents:

$$\epsilon^{jx} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \cdots \tag{70}$$

$$\epsilon^{-jx} = 1 - jx - \frac{x^2}{2!} + j\frac{x^3}{3!} + \cdots \tag{71}$$

may be derived the relation between the trigonometric and the exponential functions, namely,

$$\sin x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2j} \tag{72}$$

$$\cos x = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2} \tag{73}$$

It may be of interest to compare these important equations with equations (62) and (63). From equations (72) and (73), the following useful relation may be obtained:

$$\epsilon^{\pm jx} = \cos x \pm j \sin x \tag{74}$$

From equations (62) and (63) may be derived

$$\cosh^2 x - \sinh^2 x = 1 \tag{75}$$

which may be compared with the well-known

$$\sin^2 x + \cos^2 x = 1 \tag{76}$$

By making use of equations (62) and (63) in conjunction with equation (74) a series of very important and most useful formulas may be obtained. A few of these are listed below.

$$\sinh 2x = 2 \sinh x \cosh x \tag{77}$$

$$\begin{aligned}\cosh 2x &= \sinh^2 x + \cosh^2 x \\ &= 2 \sinh^2 x + 1 = 2 \cosh^2 x - 1\end{aligned} \tag{78}$$

$$2 \sinh^2 \frac{x}{2} = \cosh x - 1 \tag{79}$$

$$2 \cosh^2 \frac{x}{2} = \cosh x + 1 \tag{80}$$

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{81}$$

$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{82}$$

$$\tanh (x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \tag{83}$$

$$\sinh jy = j \sin y \tag{84}$$

$$\cosh jy = \cos y \tag{85}$$

$$\tanh jy = j \tan y \tag{86}$$

$$\sinh (x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y \tag{87}$$

$$\cosh (x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y \tag{88}$$

$$\sinh (x \pm j2\pi) = \sinh x \tag{89}$$

$$\sinh (x \pm j\pi) = -\sinh x \tag{90}$$

$$\sinh \left(x \pm j\frac{\pi}{2}\right) = \pm j \cosh x \tag{91}$$

$$\cosh (x \pm j2\pi) = \cosh x \tag{92}$$

$$\cosh (x \pm j\pi) = -\cosh x \tag{93}$$

$$\cosh \left(x \pm j\frac{\pi}{2}\right) = \pm j \sinh x \tag{94}$$

$$\sinh [x \pm j(y \pm 2\pi)] = \sinh (x \pm jy) \tag{95}$$

$$\sinh [x \pm j(y \pm \pi)] = -\sinh(x \pm jy) \tag{96}$$

$$\sinh \left[x \pm j\left(y \pm \frac{\pi}{2}\right)\right] = \pm j \cosh (x \pm jy) \tag{97}$$

$$\cosh [x \pm j(y \pm 2\pi)] = \cosh (x \pm jy) \tag{98}$$

$$\cosh [x \pm j(y \pm \pi)] = -\cosh (x \pm jy) \tag{99}$$

$$\cosh \left[x \pm j\left(y \pm \frac{\pi}{2}\right)\right] = \pm j \sinh (x \pm jy) \tag{100}$$

$$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} \tag{101}$$

$$\sinh x = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} \tag{102}$$

$$\cosh x = \frac{1}{\sqrt{1 - \tanh^2 x}} \tag{103}$$

$$\sinh x = \frac{2 \tanh \frac{x}{2}}{1 + \tanh^2 \frac{x}{2}} \tag{104}$$

All of these relations can easily be proved. For instance, equation (87)

$$\begin{aligned}
\sinh (x \pm jy) &= \frac{\epsilon^{(x \pm jy)} - \epsilon^{-(x \pm jy)}}{2} \\
&= \tfrac{1}{2}[\epsilon^x \epsilon^{\pm jy} - \epsilon^{-x} \epsilon^{\mp jy}] \\
&= \tfrac{1}{2}[\epsilon^x(\cos y \pm j \sin y) - \epsilon^{-x}(\cos y \mp j \sin y)] \\
&= \frac{\epsilon^x - \epsilon^{-x}}{2} \cos y \pm j \frac{\epsilon^x + \epsilon^{-x}}{2} \sin y \\
&= \sinh x \cos y \pm j \cosh x \sin y
\end{aligned}$$

In determining the position angle of the load, it is often convenient to make use of a formula which will be developed below. First, however, it is necessary to derive a few auxiliary relations which then will be used in obtaining the desired formula.

The logarithm of a complex quantity is given by

$$\begin{aligned}
\log (a \pm jb) &= \log \left[\sqrt{a^2 + b^2}\, \epsilon^{\pm j \tan^{-1} \frac{b}{a}} \right] \\
&= \log \sqrt{a^2 + b^2} \pm j \tan^{-1} \frac{b}{a}
\end{aligned} \tag{105}$$

In general, the angle whose hyperbolic tangent is u (where u may be real or complex) may be written

$$\tanh^{-1} u = \frac{1}{2} \log \frac{1 + u}{1 - u} \tag{106}$$

This relation is readily proved as follows: If

$$x = \tanh^{-1} u$$

then

$$u = \tanh x = \frac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}} = \frac{\epsilon^{2x} - 1}{\epsilon^{2x} + 1}$$

from which

$$\epsilon^{2x} = \frac{1 + \tanh x}{1 - \tanh x} = \frac{1 + u}{1 - u}$$

Taking logarithms gives

$$2x = \log \frac{1 + u}{1 - u}$$

from which equation (106) immediately follows.

Now, the position angle of a load impedance is the angle whose hyperbolic tangent is the ratio of the load impedance to the surge impedance (see equation (23)). In general, this ratio is complex and of the form $a \pm jb$, making it necessary to evaluate $\tanh^{-1}(a \pm jb)$. Making use of equation (106) gives

$$\tanh^{-1}(a \pm jb) = \tfrac{1}{2}\log\frac{1 + a \pm jb}{1 - a \mp jb}$$
$$= \tfrac{1}{2}[\log(1 + a \pm jb) - \log(1 - a \mp jb)]$$

The complex logarithms in this expression can be taken care of by equation (105). Hence,

$$\tanh^{-1}(a \pm jb) =$$
$$\frac{1}{2}\left[\log\sqrt{(1+a)^2 + b^2} \pm j\tan^{-1}\frac{b}{1+a} - \log\sqrt{(1-a)^2 + b^2} \pm j\tan^{-1}\frac{b}{1-a}\right]$$
$$= \frac{1}{2}\log\sqrt{\frac{(1+a)^2 + b^2}{(1-a)^2 + b^2}} \pm j\,\frac{\tan^{-1}\dfrac{b}{1+a} + \tan^{-1}\dfrac{b}{1-a}}{2} \qquad (107)$$

The second term in this formula may be modified, giving

$$\tanh^{-1}(a + jb) = \frac{1}{2}\log\sqrt{\frac{(1+a)^2 + b^2}{(1-a)^2 + b^2}} \pm \frac{\pi - \tan^{-1}\dfrac{1+a}{b} - \tan^{-1}\dfrac{1-a}{b}}{2} \qquad (108)$$

In applying one of these formulas (equations (107) or (108)), the one should be chosen which, for the particular numerical values of a and b, will give the arc-tangents in the numerator of the second term with the greatest precision.

Numerical Values of Hyperbolic Functions.—It is evident that the numerical values of hyperbolic functions may always be determined by resorting to the equivalent exponential formulas or by using the hyperbolic series. These methods will work equally well whether the hyperbolic angle is real or complex. Of course, evaluation of the hyperbolic functions in this manner is laborious, although any desired degree of accuracy can be obtained. Several other methods which are more practical, however, are available.

If the argument is real, the hyperbolic function can immediately be selected from tables. Probably the most complete and most accurate table is contained in the Smithsonian Mathemati-

cal Tables.[1] The values in this table are carried out to five significant places. The Smithsonian Mathematical Tables also contain a very complete table of the exponential functions carried out to seven places. From this table, the hyperbolic functions may obviously be calculated if this degree of precision is desired. Shorter tables will be found in most handbooks, for instance in Pender's "Handbook for Electrical Engineers" and in Hudson's "The Engineer's Manual," in which the hyperbolic functions are given to four significant places.

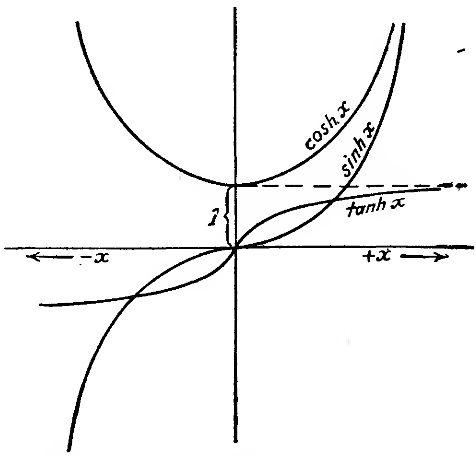

FIG. 100.—The curves show the hyperbolic sine, cosine, and tangent of real arguments.

Figure 100 shows graphically the values of the hyperbolic sine, cosine, and tangent of real arguments. It will be noted that the value of the sine passes through zero and extends to plus and minus infinity when the arguments become infinite. The value of the cosine is always positive. It is unity when the argument is zero and increases in a positive direction for increasing positive as well as negative arguments, finally reaching the value of infinity. It should be noted that the value of the cosine approaches the value of the sine for large positive arguments. The value of the tangent, which always is given by the ratio of the sine to the cosine, goes through zero and reaches a maximum value of plus and minus unity corresponding to infinite arguments.

[1] Published by the Smithsonian Institution, Washington, D. C., 1909. Reprinted 1920.

Complex hyperbolic functions may be determined by Dr. Kennelly's tables and charts.[1] His tables give the values of the complex hyperbolic sines, cosines, and tangents, and also the correction factors necessary in converting a smooth line to its equivalent T- or Π-circuit. There are tables with the values in polar as well as in rectangular form. In using the tables it is usually necessary to interpolate in two dimensions, a process which requires quite a little care and time. Straight-line interpolation as ordinarily used may, in certain cases, lead to error. Dr. Kennelly, however, outlines methods for more accurate interpolation based on Taylor's theorem which may be used when great accuracy is required.

Dr. Kennelly's charts are constructed from the values in his tables and give the same information in an extremely convenient form for engineering use. The value of any hyperbolic function can be read immediately by spotting the argument in the proper chart. Conversely, the hyperbolic angle corresponding to a given function is just as readily determined. The values, from the charts, can, as a rule, be obtained to three significant figures and are, hence, accurate enough for most engineering purposes.

It should be noted that Dr. Kennelly, in most of his tables and charts, introduces the use of quadrants instead of radians for the imaginary part of the hyperbolic angles. One quadrant is equivalent to $\pi/2$ radians. In converting from radians to quadrants, therefore, the imaginary part is multiplied by $2/\pi$, while in converting from quadrants to radians the imaginary part is multiplied by $\pi/2$. It has been previously demonstrated that the hyperbolic functions repeat themselves or change according to certain rules, when $\pi/2$, π, or 2π, respectively, are added to or subtracted from the imaginary part of the argument. It is important to keep this fact in mind and make use of it in obtaining numerical values. To do this is much easier when the imaginary part is kept in terms of quadrants, since the changes occur at *integral multiples of one quadrant.* If, on the other hand, the imaginary part is given in terms of radians, it is necessary to

[1] KENNELLY, A. E., "Tables of Complex Hyperbolic and Circular Functions." Harvard University Press, Cambridge, Mass., 1914. Reprinted 1921.

KENNELLY, A. E., "Chart Atlas of Complex Hyperbolic and Circular Functions," Harvard University Press, Cambridge, Mass., 1914. Reprinted 1921 and 1924.

keep track of *integral multiples of π, or* 3.1416 *radians,* which, quite obviously, is considerably less convenient.

If a very high degree of precision is required, the complex hyperbolic functions may be calculated from the series (equations (66) and (67)) or by equations (87) and (88). The latter (equations (87) and (88)) involve hyperbolic and trigonometric functions of real arguments only, which, when the proper tables are used, can be obtained to a large number of significant figures.

EXAMPLE 1

This example shows the application of the hyperbolic theory to the computation of voltage, current, and power distribution on an alternating-current line. It also serves to give an idea of the peculiarities of such a line when electrically very long.

Statement of Problem

A three-phase, 220-kv., 60-cycle transmission line is fed from alternators whose voltage waves contain tooth harmonics of twenty-nine times the fundamental frequency. The magnitude of the tooth-harmonic voltages is 0.5 per cent of the fundamental.

The constants of the line are

$$\text{Length} = 250 \text{ miles}$$

$$\left.\begin{aligned} r &= 0.127 \text{ ohm} \\ g &= 1.47 \times 10^{-8} \text{ mho} \\ L &= 0.00212 \text{ henry} \\ C &= 0.0142 \times 10^{-6} \text{ farad} \end{aligned}\right\} \quad \text{per wire mile}$$

Compute and plot the distribution of voltage, current, vector power, and active power of tooth-harmonic frequency when the distant end of the line is *free.*

Solution

The values of the hyperbolic functions used below are all obtained from Dr. Kennelly's charts.

$$\omega = 2\pi 60 \times 29 = 10{,}930 \text{ rad./sec.}$$

$$z = r + j\omega L = 0.127 + j10{,}930 \times 0.00212$$

$$= 0.127 + j23.18 = 23.18\underline{/89^\circ.68} \text{ ohms/mile}$$

$$y = g + j\omega C = (1.47 + j10{,}930 \times 10^2 \times 0.0142) \times 10^{-8}$$

$$= (1.47 + j15{,}520)10^{-8} = 155.2 \times 10^{-6}\underline{/89^\circ.92} \text{ mho/mile}$$

$$\alpha = \sqrt{zy} = \sqrt{23.18\underline{/89^\circ.68} \times 155.2 \times 10^{-6}\underline{/89^\circ.92}}$$

$$= 10^{-3}\sqrt{3{,}597\underline{/89^\circ.8}} = 0.060\underline{/89^\circ.8} \text{ hyp/mile}$$

$$= 0.00021 + j0.060 = 0.00021 + \underline{j0.0382} \text{ hyp/mile}$$

$$\theta = 250 \times \alpha = 0.0525 + \underline{j9.55} \text{ hyp}$$

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{23.18\underline{/89^\circ.68}}{155.2 \times 10^{-6}\underline{/89^\circ.92}}} = 10^3\sqrt{0.1492}\,\overline{\backslash 0^\circ.12}$$

$$= 386.2\overline{\backslash 0^\circ.12} \text{ ohms}$$

TABLE IX.—CALCULATION OF VOLTAGE, CURRENT, AND POWER DISTRIBUTION AT TOOTH-HARMONIC FREQUENCY ON A LONG TRANSMISSION LINE, THE DISTANT END BEING FREE (EXAMPLE 1)

Miles from receiving end	θ_x, hyperbolic radians	sinh θ_x	cosh θ_x	V_x, volts	I_x, amperes	Vector power referred to voltage	Active power, watts
0	0 + j0	0/90°	1.000 /0	834 \897°.4	0 \807°.3	0/90°	0
25	0.00525 + j0.955	1.000/89°.9	0.062 /4°.0	51.7\893°.4	2.16 \807°.4	111.9/86°.0	7.61
26.2	0.0055 + j1.000	1.000/90°.0	0.0055/90°	4.6\807°.4	2.16 \807°.3		
50	0.0105 + j1.910	0.142/94°.0	0.991 /179°.9	826 \717°.5	0.307 \803°.3	253 \85°.8	18.51
52.3	0.0110 + j2.000	0.011/180°.0	1.000 /180°.0	834 \717°.4	0.0238\717°.3		
75	0.0158 + j2.865	0.976/269°.8	0.211 /184°.2	176 \713°.2	2.11 \627°.5	371 /85°.7	27.85
78.5	0.0165 + j3.000	1.000/270°.0	0.0165/270°.0	13.8\627°.4	2.16 \627°.3		
100	0.0210 + j3.820	0.280/274°.0	0.958 /359°.8	799 \537°.6	0.605 \623°.3	483 \85°.7	36.3
104.7	0.0220 + j4.000	0.022/360°.0	1.000 /360°.0	834 \537°.4	0.0475\537°.3		
125	0.0263 + j4.775	0.938/449°.5	0.346 /364°.0	289 \533°.4	2.11 \447°.8	586 /85°.6	45.0
131	0.0275 + j5.000	1.000/450°.0	0.0275/450°.0	22.9\447°.4	2.16 \447°.3		
150	0.0315 + j5.730	0.415/453°.8	0.912 /539°.1	761 \358°.3	0.897 \443°.5	684 \85°.2	57.3
157.1	0.0330 + j6.000	0.033/540°.0	1.001 /540°.0	835 \357°.4	0.0713\357°.3		
175	0.0368 + j6.685	0.879/628°.9	0.476 /543°.8	397 \353°.6	1.898 \268°.4	754 /85°.2	63.0
183.3	0.0385 + j7.000	1.001/630°.0	0.0385/630°.0	32.1\267°.4	2.17 \267°.3		
200	0.0420 + j7.640	0.538/633°.8	0.845 /718°.5	705 \178°.9	1.163 \263°.5	820 \84°.6	77.2
209.3	0.0440 + j8.000	0.044/720°.0	1.001 /720°.0	835 \177°.4	0.0951\177°.3		
225	0.0473 + j8.595	0.804/808°.2	0.601 /723°.7	502 \173°.7	1.738 \89°.1	872 /84°.6	82.0
235.7	0.0495 + j9.000	1.001/810°.0	0.0495/810°.0	41.3\87°.4	2.17 \87°.3		
250	0.0525 + j9.550	0.650/813°.5	0.761 /897°.4	635 \0	1.405 \83°.8	891 \83°.8	96.2

$$Z_s = Z_0 \coth \theta = 386.2\overline{\backslash 0°.1}\,\frac{0.761\underline{/897°.4}}{0.650\underline{/813°.5}} = 452.5\underline{/83°.8} \text{ ohms}$$

$$E_s = \frac{220{,}000 \times 0.005}{\sqrt{3}} = \frac{1{,}100}{\sqrt{3}} = 635 \text{ volts (of tooth-harmonic frequency)}$$

$$I_s = \frac{E_s}{Z_s} = \frac{635\underline{/0}}{452.5\underline{/83°.8}} = 1.405\overline{\backslash 83°.8} \text{ amp.}$$

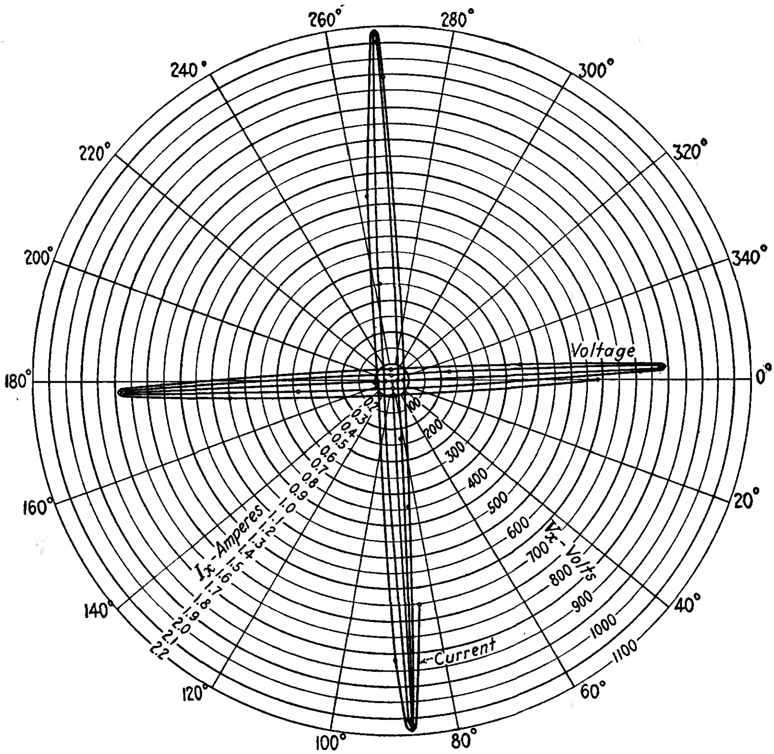

FIG. 101.—Voltage and current distribution on an electrically long transmission line, whose distant end is open, plotted in polar coordinates. The voltage at the sending end is used as standard phase (for calculations, see Example 1).

The voltage and current distribution are given by

$$E_x = E_s \frac{\cosh \theta_x}{\cosh \theta} \tag{a}$$

$$I_x = I_s \frac{\sinh \theta_x}{\sinh \theta} \tag{b}$$

Vector power

$$P_x = E_x\underline{/0} \times I_x\underline{/\phi} \text{ volt-amp.} \tag{c}$$

where ϕ is referred to $E_x\underline{/0}$.

Active power

$$P_x = E_x I_x \cos \phi \text{ watts} \tag{d}$$

The numerical calculations of the voltage, current, and power distribution are given in Table IX. It will be noted that points spaced regularly along the line 25 miles apart have been used and, in addition, a certain number of points so selected that the hyperbolic angle subtended at these points have imaginary parts representing an *integral number* of *quadrants*. It is advisable in calculations of this sort to make use of these "quadrantal points" in order to benefit by the greater ease with which the hyperbolic functions may be determined at these points.

The voltage and current distribution have been plotted in Fig. 101 in polar coordinates, *i.e.*, *versus* angular displacement with reference to the sending voltage (of tooth-harmonic frequency). This diagram shows strikingly

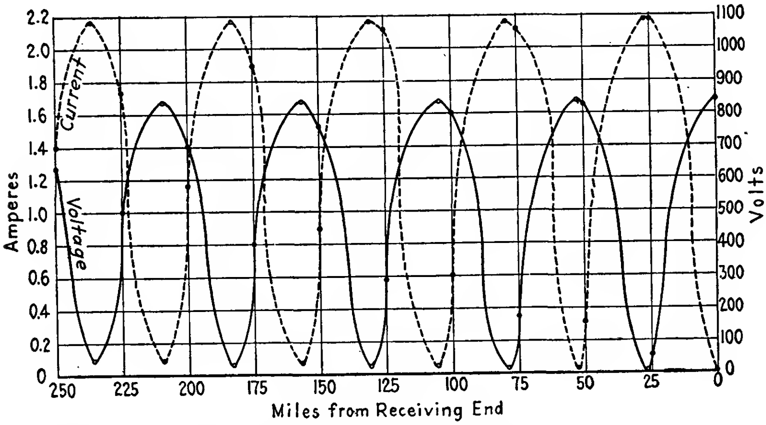

FIG. 102.—Voltage and current distribution along an electrically long transmission line, whose distant end is open, plotted *versus* distance measured along the line (for calculations, see Example 1).

the distributions which are characteristic of a line which is electrically very long and whose distant end is free. The spirals which, in this case, as seen, are very narrow, broaden out when load is applied at the receiver end of the line. In no case, however, will crossing occur of lines belonging to the same spiral.

Figure 102 shows the voltage and current distribution plotted in rectangular coordinate *versus* distance measured along the line. This diagram gives a good picture of the very great variation in voltage and current which occurs from point to point on an electrically long line. It may be noted that both voltage and current go through five maximum and five minimum values. This will be the case whenever the line subtends a hyperbolic angle whose imaginary part lies between nine and ten quadrants.

The wave length (as previously defined) corresponding to tooth-harmonic frequency on this line would be

$$\lambda = \frac{2\pi}{\alpha_2} = \frac{2\pi}{0.060} = 104.8 \text{ miles}$$

Hence, the line is actually $\frac{250}{104.8} = 2.385$ wave lengths long.

The distributions of vector power and active power are plotted *versus* distance in Fig. 103. The vector power shows a smooth variation from its maximum value at the sending end to zero at the receiving end. The active power on the other hand is of a "pulsating" nature. It should be noted, however, that the active power continually decreases along the line. Obviously this must always be the case.

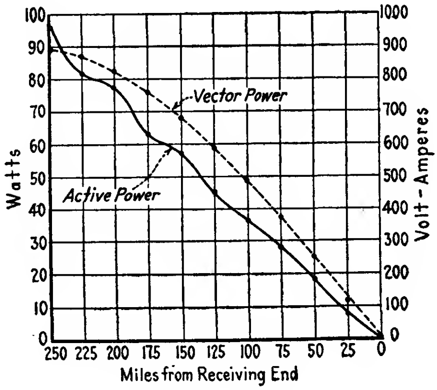

FIG. 103.—Distribution of vector power and active power along an electrically long transmission line, whose distant end is open, plotted *versus* distance measured along the line (for calculations, see Example 1).

EXAMPLE 2

Statement of Problem

The line described in Example 1 is delivering 60,000 kw. at 80 per cent power factor (lagging). The receiver voltage is strictly 220 kv. Determine the position angle of this load by application of equation (108).

Solution

If ϕ designates the power-factor angle, the load impedance is given by

$$\sigma = \frac{E_r}{I_r}\underline{/\phi} = \frac{220{,}000/\sqrt{3}}{\dfrac{60{,}000}{\sqrt{3} \times 220 \times 0.8}}\underline{/\cos^{-1} 0.8}$$

$$= \frac{220^2 \times 0.8}{60}\underline{/\cos^{-1} 0.8} = 645.3\underline{/36^\circ.86} \text{ ohms}$$

The linear impedance and admittance of the line at 60 cycles per second become

$$z = r + j\omega L = 0.127 + j377 \times 0.00212$$
$$= 0.127 + j0.799 = 0.809\underline{/80^\circ.96} \text{ ohm}$$
$$y = g + j\omega C = (1.47 + j377 \times 0.0142 \times 10^2)10^{-8}$$
$$= (1.47 + j535)10^{-8} = 5.35 \times 10^{-6}\underline{/89^\circ.85} \text{ mhos}$$

The 60-cycle surge impedance of the line is, hence,

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{0.809\underline{/80^\circ.96}}{5.35 \times 10^{-6}\underline{/89^\circ.85}}} = 388.5\overline{\backslash 4^\circ.45} \text{ ohms}$$

$$\frac{\sigma}{Z_0} = \frac{645.3/36°.86}{388.5\backslash 4°.45} = 1.660/41°.31 = 1.246 + j1.096$$

The angle of the load θ' is given by

$$\theta' = \tanh^{-1}\frac{\sigma}{Z_0} = \tan^{-1}(1.246 + j1.096)$$

$$= \frac{1}{2}\log\sqrt{\frac{2.246^2 + 1.096^2}{0.246^2 + 1.096^2}} + j\frac{\pi - \tan^{-1}\frac{2.246}{1.096} - \tan^{-1}\frac{-0.246}{1.096}}{2}$$

$$= \frac{1}{2}\log\sqrt{\frac{6.251}{1.262}} + j\frac{\pi - \tan^{-1}2.050 - \tan^{-1}(-0.2243)}{2}$$

$$= \frac{1}{2}\log 2.224 + j\frac{\pi - 1.1175 + 0.2204}{2} = 0.400 + j1.122 \text{ hyp}$$

Above, the angle corresponding to a negative tangent was considered negative and less than $\pi/2$ radians. This should always be done if the angle corresponding to the positive tangent is taken in the first quadrant, which presumably usually will be the case. It might be noted, however, that the two angles may be taken in the third and second quadrants, respectively, both being considered positive. The results would then be

$$\theta' = 0.400 - j2.019 \text{ hyp}$$

It is seen that the imaginary parts of the two solutions are displaced by π radians. Hence, the second result is obviously also correct, since the values of the hyperbolic tangents repeat themselves when a multiple of π is added to or subtracted from the imaginary part of the hyperbolic angle.

It is immaterial which one of the load position angles above is used in computing the voltage, current, and power distribution on the line. Evidently, the imaginary part of the position angle at any point on the line will differ by π radians in the two cases. This does not matter in the least, however. In the first place, such a change (or a change by a multiple of π) does not change the hyperbolic tangents and, hence, the values of impedance (including sending-end impedance) which all depend on these tangents. Secondly, although hyperbolic sines and cosines change sign when the imaginary part of the angles is increased or decreased by π (or a multiple of π) radians, the computed voltage, current, and power distributions will still be correct since they depend on ratios of hyperbolic sines and cosines. Consequently, any effect of the reversal in signs is eliminated.

Direct-current Lines.—As previously stated, the direct-current problem may be looked upon as a special case of the alternating-current problem. Hence, the formulas which have been derived are all applicable to direct current when the frequency is considered zero. In general, therefore, the direct-current problem is simpler, since, in the steady state, the effect of inductance and capacitance is eliminated, resistance and leakance only having to be considered. Furthermore, all quantities involved, including the hyperbolic functions, become real rather than complex.

Thus, for a direct-current line, the attenuation constant is real and may be written

$$\alpha = \sqrt{rg} \tag{109}$$

The surge impedance, which now appropriately may be termed the *surge resistance*, becomes

$$R_0 = \sqrt{\frac{r}{g}} \tag{110}$$

It is always possible to solve any problem on distribution of voltage, current, and power along a direct-current line by using real hyperbolic functions only. In this connection, however, one particular case deserves special mention. Assume the line to be loaded with a resistance larger than the surge resistance. The hyperbolic angle of the load then becomes

$$\theta' = \tanh^{-1}\frac{\sigma}{R_0} = \tanh^{-1}(\text{real number} > 1) \tag{111}$$

No such angle exists in the real region, the maximum possible value of the hyperbolic tangent of a real angle being 1. Equation (111), however, may be written

$$\theta' = \tanh^{-1}\frac{R_0}{\sigma} + j\frac{\pi}{2} = \theta'' + j\frac{\pi}{2} \tag{112}$$

This is easily verified, since, in general,

$$\tanh\theta' = \frac{\sigma}{R_0} = \coth\left(\theta' \pm j\frac{\pi}{2}\right)$$

Hence,

$$\theta' \pm j\frac{\pi}{2} = \coth^{-1}\frac{\sigma}{R_0} = \tanh^{-1}\frac{R_0}{\sigma}$$

from which equation (112) immediately follows. Evidently $\theta'' = \tanh^{-1} R_0/\sigma$ is always obtainable from a table of real hyperbolic functions when $\sigma > R_0$.

The voltage and current distribution is now given by equations (28) and (29)

$$E_x = E_s\frac{\sinh\left(\theta_x + \theta'' \pm j\frac{\pi}{2}\right)}{\sinh\left(\theta + \theta'' \pm j\frac{\pi}{2}\right)} = E_s\frac{\cosh(\theta_x + \theta'')}{\cosh(\theta + \theta'')} \tag{113}$$

$$I_x = I_s\frac{\cosh\left(\theta_x + \theta'' \pm j\frac{\pi}{2}\right)}{\cosh\left(\theta + \theta'' \pm j\frac{\pi}{2}\right)} = I_s\frac{\sinh(\theta_x + \theta'')}{\sinh(\theta + \theta'')} \tag{114}$$

The sending-end impedance becomes (from equation (32))

$$Z_s = Z_0 \tanh\left(\theta + \theta'' \pm j\frac{\pi}{2}\right) = Z_0 \coth(\theta + \theta'') \qquad (115)$$

Since θ_x, θ, and θ'' are real quantities, equations (113), (114), and (115) can all be evaluated by means of real hyperbolic functions.

EXAMPLE 3

Statement of Problem

A ground-return telegraph line, 320 km. long, has the following constants:

$$r = 8.5 \text{ ohms/km.}$$
$$g = 2.4 \times 10^{-6} \text{ mho/km.}$$

Calculate and plot the voltage and current distribution on this line when 1 volt direct current is impressed at the sending end and the load resistance is $\sigma = 6{,}430$ ohms.

Solution

The attenuation constant and surge resistance are

$$\alpha = \sqrt{rg} = \sqrt{8.5 \times 2.4 \times 10^{-6}} = 0.00452 \text{ hyp/km.}$$

$$R_0 = \sqrt{\frac{r}{g}} = \sqrt{\frac{8.5}{2.4 \times 10^{-6}}} = 1{,}881 \text{ ohms}$$

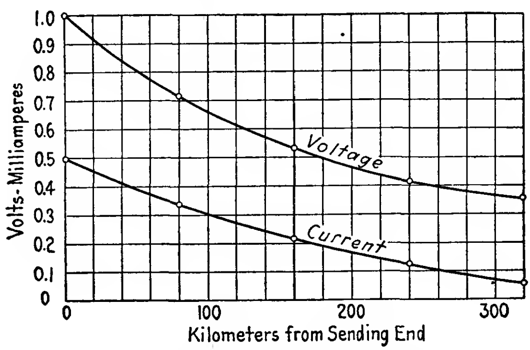

FIG. 104.—Voltage and current distribution along a ground-return telegraph line. One volt direct current is impressed at the sending end, the line being loaded with a resistance of 6,430 ohms at the receiving end (for calculations, see Example 3).

It is seen that, in this case, the load resistance is larger than the surge resistance. Hence, equations (113), (114), and (115) should be used.

$$\theta'' = \tanh^{-1}\frac{R_0}{\sigma} = \tanh^{-1}\frac{1{,}881}{6{,}430} = \tanh^{-1} 0.293 = 0.303 \text{ hyp}$$

The entering current becomes

$$I_s = \frac{E_s \tanh(\theta + \theta'')}{R_0} = \frac{\tanh 1.7494}{1{,}881} = \frac{0.9413}{1{,}881} = 0.500 \times 10^{-3} \text{ amp.}$$

Table X gives the calculations of the voltage and current distribution, the distance x being measured from the receiver end. Figure 104 shows the shapes of the voltage and current distributions on the line.

TABLE X.—CALCULATION OF DIRECT-CURRENT VOLTAGE AND CURRENT DISTRIBUTION ON A TELEGRAPH LINE, THE LOAD RESISTANCE BEING LARGER THAN THE SURGE RESISTANCE (EXAMPLE 3)

x kilometers	$\alpha x = \theta_x$ hyperbolic radians	θ'' hyperbolic radians	$\theta_x + \theta''$ hyperbolic radians	$\cosh(\theta_x + \theta')$	$\sinh(\theta_x + \theta'')$	E_x, volts	I_x, milliamperes
0	0	0.303	0.3030	1.0462	0.3077	0.353	0.055
80	0.3616	0.303	0.6646	1.2291	0.7147	0.415	0.128
160	0.7232	0.303	1.0262	1.5744	1.2160	0.532	0.218
240	1.0848	0.303	1.3878	2.1279	1.8782	0.718	0.337
320	1.4464	0.303	1.7494	2.9625	2.7886	1.000	0.500

EXAMPLE 4

Statement of Problem

A ground-return telegraph line (Fig. 105) l miles in total length has in series with it at each end a battery and a relay. The voltages of the bat-

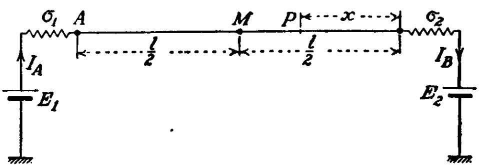

FIG. 105.—Ground-return telegraph line with batteries and relays at each end (Example 4).

teries are E_1 and E_2 volts, respectively, and the resistances of the relays are σ_1 and σ_2 ohms. The line has a resistance of r ohms per mile and a conductance to ground of g mhos per mile.

1. Establish a general expression for the current through the relays, the current at the middle of the line, and the voltage above ground at the middle of the line. Treat both the case where the batteries act in conjunction and the case where the batteries act in opposition.

2. Introducing the following numerical values:

$$l = 90 \text{ miles}$$
$$E_1 = E_2 = 100 \text{ volts}$$
$$\sigma_1 = \sigma_2 = 400 \text{ ohms}$$
$$r = 20 \text{ ohms per mile}$$
$$g = 5 \times 10^{-6} \text{ mhos per mile}$$

compute the relay currents, the current at the midpoint, and the voltage at the midpoint for the case where the batteries act in conjunction.

Solution

1. *Derivation of Formulas:*

The voltage and current at the point P are

$$V_X = V_B \cosh \alpha x + I_B R_0 \sinh \alpha x \qquad (a)$$

$$I_X = I_B \cosh \alpha x + \frac{V_B}{R_0} \sinh \alpha x \qquad (b)$$

where

$$R_0 = \sqrt{\frac{r}{g}}, \quad \alpha = \sqrt{rg}$$

The voltages at A and B are also given by

$$V_A = E_1 - \sigma_1 I_A \tag{c}$$

$$V_B = (\pm E_2) + \sigma_2 I_B \tag{d}$$

where the plus sign applies when E_2 acts in opposition to E_1, the minus sign when it acts in conjunction with E_1.

V_A and I_A can be found from equations (a) and (b) by substituting $x = l$. Hence,

$$V_A = E_1 - \sigma_1 I_A = (\pm E_2 - \sigma_2 I_B) \cosh \alpha l + I_B R_0 \sinh \alpha l \tag{e}$$

$$I_A = I_B \cosh \alpha l + \frac{(\pm E_2 - \sigma_2 I_B)}{R_0} \sinh \alpha l \tag{f}$$

Solving equations (e) and (f) for I_A and I_B gives

$$I_A = \frac{E_1\left(\cosh \alpha l + \frac{\sigma_2}{R_0} \sinh \alpha l\right) - (\pm E_2)}{(\sigma_1 + \sigma_2) \cosh \alpha l + \left(\frac{\sigma_1 \sigma_2}{R_0} + R_0\right) \sinh \alpha l} \tag{g}$$

$$I_B = \frac{E_1 - (\pm E_2)\left(\cosh \alpha l + \frac{\sigma_1}{R_0} \sinh \alpha l\right)}{(\sigma_1 + \sigma_2) \cosh \alpha l + \left(\frac{\sigma_1 \sigma_2}{R_0} + R_0\right) \sinh \alpha l} \tag{h}$$

The current at the middle point can be obtained from equation (b) by substituting $x = l/2$ giving

$$I_M = I_B \cosh \frac{\alpha l}{2} + \frac{V_B}{R_0} \sinh \frac{\alpha l}{2} \tag{i}$$

Multiplying equation (i) through by $\cosh \alpha l$ gives

$$\begin{aligned} I_M \cosh \frac{\alpha l}{2} &= I_B \cosh^2 \frac{\alpha l}{2} + \frac{V_B}{R_0} \sinh \frac{\alpha l}{2} \cosh \frac{\alpha l}{2} \\ &= \frac{I_B}{2}(\cosh \alpha l + 1) + \frac{V_B}{2R_0} \sinh \alpha l \\ &= \frac{I_B}{2} + \frac{1}{2}\left(I_B \cosh \alpha l + \frac{V_B}{R_0} \sinh \alpha l\right) = \frac{I_B + I_A}{2} \end{aligned}$$

Hence,

$$I_M = \frac{I_B + I_A}{2 \cosh \frac{\alpha l}{2}} \tag{j}$$

Substituting equations (g) and (h) in equation (j) gives for the current at the midpoint

$$I_M = \frac{1}{2 \cosh \frac{\alpha l}{2}} \frac{(E_1 - \pm E_2) + (E_1 - \pm E_2) \cosh \alpha l + \frac{\sigma_1}{R_0} \sinh \alpha l (E_1 - \pm E_2)}{(\sigma_1 + \sigma_2) \cosh \alpha l + \left(\frac{\sigma_1 \sigma_2}{R_0} + R_0\right) \sinh \alpha l} \tag{k}$$

As before, the upper sign (+) is for opposition of E_1 and E_2, the lower sign (−) for conjunction.

By a procedure similar to the one used above in deriving I_M, the following expression for the voltage at the midpoint is obtained:

$$V_M = \frac{V_A + V_B}{2 \cosh \frac{\alpha l}{2}} \tag{l}$$

and also,

$$V_M = \frac{1}{2\cosh \frac{\alpha l}{2}}[(E_1 + \pm E_2) + \sigma_2 I_B - \sigma_1 I_A] = \frac{1}{2\cosh \frac{\alpha l}{2}}$$

$$\left[(E_1 + \pm E_2) + \frac{(E_1\sigma_2 \pm E_2\sigma_1) - (E_1\sigma_1 \pm E_2\sigma_2)\cosh \alpha l - (E_1 \pm E_2)\frac{\sigma_1\sigma_2}{R_0}\sinh \alpha l}{(\sigma_1 + \sigma_2)\cosh \alpha l + \left(\frac{\sigma_1\sigma_2}{R_0} + R_0\right)\sinh \alpha l}\right] \tag{m}$$

2. *Numerical Computations:*

$$R_0 = \sqrt{\frac{r}{g}} = \sqrt{\frac{20}{5 \times 10^{-6}}} = 2{,}000 \text{ ohms}$$

$$\alpha l = l\sqrt{rg} = 90\sqrt{20 \times 5 \times 10^{-6}} = 0.90 \text{ hyp}$$

$$\cosh \alpha l = 1.4331$$

$$\sinh \alpha l = 1.0265$$

$$\cosh \frac{\alpha l}{2} = 1.1030$$

The value of the denominators in equations (g) and (h) becomes

$$(\sigma_1 + \sigma_2)\cosh \alpha l + \left(\frac{\sigma_1\sigma_2}{R_0} + R_0 \sinh \alpha l\right) = 800 \times 1.4331 + 2{,}080 \times 1.0265 = 3{,}280.0$$

Since the batteries act in conjunction, the minus sign for E_2 should be used. Hence, from equation (g),

$$I_A = \frac{100\left(\cosh \alpha l + \frac{400}{2{,}000}\sinh \alpha l\right) + 100}{3{,}280} = \frac{263.82}{3{,}280} = 0.0804 \text{ amp.}$$

From equation (h)

$$I_B = 0.0804 \text{ amp.}$$

From equation (j)

$$I_M = \frac{I_A + I_B}{2\cosh \frac{\alpha l}{2}} = \frac{0.0804}{\cosh \frac{\alpha l}{2}} = 0.0730 \text{ amp.}$$

From equation (m)

$$V_M = 0$$

NOTE.—In the particular case of equal battery voltages and equal resistances of the relays, *i.e.*, where complete symmetry exists with respect to the midpoint, the problem can be readily solved without resorting to the general expressions derived above. It is evident by inspection that, when such

symmetry is present and the batteries act in conjunction, the voltage at the midpoint must of necessity be zero, while with the batteries in opposition the current at the midpoint must be zero.

Recognizing this fact, solutions may be obtained by considering the line split at the midpoint, and treating each half separately, or, still shorter, by treating just one-half of the line, since the numerical values of voltage and current will be the same in the two halves, although their directions (or signs) may be different. In the former case (batteries acting in conjunction) the half of the line considered should be assumed grounded at the distant end (corresponding to the midpoint), while in the latter case (batteries acting in opposition) the distant end should be assumed free. The solutions can then be written down immediately by making use of the formulas previously worked out for grounded and open lines.

With the batteries in conjunction, the following equations will apply:

$$I_A = I_B = \frac{E}{\sigma + R_0 \tanh \frac{\alpha l}{2}} \tag{n}$$

$$V_A = E - \sigma I_A = E\left(1 - \frac{\sigma}{\sigma + R_0 \tanh \frac{\alpha l}{2}}\right) \tag{o}$$

$$I_M = \frac{I_A}{\cosh \frac{\alpha l}{2}} = \frac{E}{\sigma \cosh \frac{\alpha l}{2} + R_0 \sinh \frac{\alpha l}{2}} \tag{p}$$

With the batteries in opposition, the following equations will apply

$$I_A = -I_B = \frac{E}{\sigma + R_0 \coth \frac{\alpha l}{2}} \tag{q}$$

$$V_A = E - \sigma I_A = E\left(1 - \frac{\sigma}{\sigma + R_0 \coth \frac{\alpha l}{2}}\right) \tag{r}$$

$$V_M = \frac{V_A}{\cosh \frac{\alpha l}{2}} = \frac{ER_0}{\sigma \sinh \frac{\alpha l}{2} + R_0 \cosh \frac{\alpha l}{2}} \tag{s}$$

Composite and Bifurcated Lines.—When there is an abrupt change in electrical constants at one or more points along a line, it is said to be *composite*. Thus, an aerial line and a cable would, together, constitute a composite line. The steady-state analysis of the composite line can be carried through without difficulty; the complete numerical solution of a problem involving composite lines, however, is, as might be expected, considerably more laborious than the solution of a simple line problem.

Consider the two-section composite line $ABCD$ shown in Fig. 106. This line is loaded at D with an impedance σ. The surge impedances are Z_{01} and Z_{02} and the total hyperbolic angles θ_1 and θ_2 of the two sections, respectively.

The position angle of the load is given by

$$\theta' = \tanh^{-1} \frac{\sigma}{Z_{02}} \tag{116}$$

At C the position angle is given by

$$\delta_C = \theta_2 + \theta' \tag{117}$$

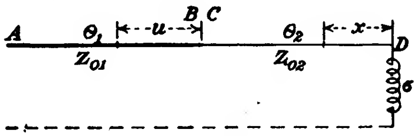

FIG. 106.—Two-section composite line with load impedance at the end.

At C there is a discontinuity in constants, but there can be no discontinuity in voltage, current, and power. The impedance at C is obviously given by

$$Z_C = Z_{02} \tanh \delta_C \tag{118}$$

This impedance may now be considered to act as a load on the first section of the line and the position angle at B determined by

$$\delta_B = \tanh^{-1} \frac{Z_C}{Z_{01}} \tag{119}$$

The position angle at the sending end A is obviously

$$\delta_A = \theta_1 + \delta_B \tag{120}$$

and the total sending-end impedance

$$Z_A = Z_{01} \tanh \delta_A \tag{121}$$

The distribution of voltage and current on the first section of the line is given by

$$E_u = E_A \frac{\sinh \delta_u}{\sinh \delta_A} \tag{122}$$

$$I_u = I_A \frac{\cosh \delta_u}{\cosh \delta_A} \tag{123}$$

where

$$\delta_u = \theta_u + \delta_B = \alpha_1 u + \delta_B \tag{124}$$

the distances u being figured from the junction point BC.

By equations (122) and (123), the voltage and current at the junction point are determined and used as reference voltage and

current in computing the distribution on the second section. Here the following equations are used:

$$E_x = E_C \frac{\sinh \delta_x}{\sinh \delta_C} \tag{125}$$

$$I_x = I_C \frac{\cosh \delta_x}{\cosh \delta_C} \tag{126}$$

where

$$\delta_x = \theta_x + \theta' = \alpha_2 x + \theta' \tag{127}$$

Evidently, by simple transformations of the equations, it is possible to use the voltage and current at D as reference in computing the distribution on section CD and the voltage and current at B in connection with section AB. It all depends on what values initially are known.

If the two-section composite line is loaded also at the junction point, as indicated in Fig. 107, the procedure is as before, except that the impedance at C and the impedance σ_1 of the load at the

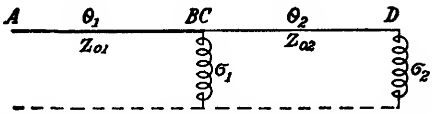

FIG. 107.—Two-section composite line with load impedance at one end and at the junction point.

junction are combined in parallel and the combination treated as a single load on the first section. Thus,

$$Z_B = \frac{\sigma_1 Z_C}{\sigma_1 + Z_C} \tag{128}$$

and

$$\delta_B = \tanh^{-1} \frac{Z_B}{Z_{01}} \tag{129}$$

In computing the distribution on the second line, it should be noted that the current at C, which may be used as reference current for this section, is obtained by subtracting the current taken by the load σ_1 from the current at B.

$$I_C = I_B - \frac{E_B}{\sigma_1} \tag{130}$$

If the composite line is bifurcated, as shown in Fig. 108, and loaded by three loads, as indicated, the impedances at C and E are first obtained by working back from D and F. These two

impedances and the load impedance σ_1 are then combined in parallel, giving the impedance at B.

$$\frac{1}{Z_B} = \frac{1}{\sigma_1} + \frac{1}{Z_C} + \frac{1}{Z_E} \tag{131}$$

By making use of the principles used for the three circuits briefly discussed above, the treatment can be extended to any arrangement of sections and any number of branches. The problem may become cumbersome, indeed, but nevertheless it is always capable of solution by straight-forward methods.

If solutions for the conditions at the terminals and at the junctions of a composite network only are desired, rather than the complete distributions on the lines themselves, perhaps the most powerful and generally applicable method is making use of the general circuit constants of the various networks and lines involved. The idea of general circuit constants, however, will not be discussed here. They are treated in detail in Chap. IX.

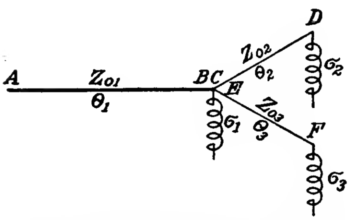

FIG. 108.—Bifurcated composite line with three impedance loads.

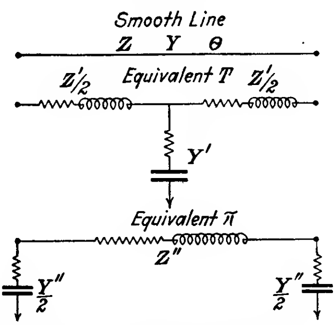

FIG. 109.—Equivalent T at π representation of a long transmission line.

Equivalent T and Π Circuits.—The nominal T and Π representation of a smooth line, previously discussed in connection with the short transmission line, become unsatisfactory when the line is electrically long. By applying correction factors, however, to the constants of the nominal T and Π, the so-called equivalent T- and Π-circuit may be obtained. These represent correctly the smooth line at *the terminals;* the equivalence, however, is limited to a *single frequency* as the correction factors depend upon the latter.

Referring to Fig. 109, let Z, Y, and θ represent the total impedance, admittance, and hyperbolic angle of the smooth line. Let $Z'/2$ and Y' represent the impedance of the arms and the admit-

tance of the staff leak, respectively, of the equivalent T and Z'' and $Y'/2$ the architrave impedance and the admittance of the pillars, respectively, of the equivalent Π. The constants of the equivalent circuits are then obtained as follows:

For T-circuit,

$$Z' = Z\frac{\tanh\frac{\theta}{2}}{\theta/2} \tag{132}$$

$$Y' = Y\frac{\sinh\theta}{\theta} \tag{133}$$

For Π-circuit,

$$Z'' = Z\frac{\sinh\theta}{\theta} \tag{134}$$

$$Y'' = Y\frac{\tanh\frac{\theta}{2}}{\theta/2} \tag{135}$$

The correctness of these formulas can readily be proved by demonstrating the actual equivalence of the smooth line and its equivalent circuits during operation as far as conditions at the terminals are concerned. It is sufficient to show that the correct general relation exists between voltage and current at the sending end and the corresponding quantities at the receiver end, and, in addition to this, that the sending-end impedance has the proper value for all conditions of loading. The proof will be given here for the T line only; the proof for the Π line may be worked in a similar manner, or simply by transformation of the T(Y)- circuit to a Π(Δ)-circuit.

The general relations between voltages and currents at the two ends of the smooth line are given by

$$E_s = E_r\cosh\theta + Z_0I_r\sinh\theta \tag{136}$$

$$I_s = I_r\cosh\theta + \frac{E_r}{Z_0}\sinh\theta \tag{137}$$

For the T line, the corresponding relations become

$$E_s = E_r\left(1 + \frac{Z'Y'}{2}\right) + I_rZ'\left(1 + \frac{Z'Y'}{4}\right) \tag{138}$$

$$I_s = I_r\left(1 + \frac{Z'Y'}{2}\right) + E_rY' \tag{139}$$

It will now be shown that equations (138) and (139) reduce to equations (136) and (137), respectively.[1]

$$1 + \frac{Z'Y'}{2} = 1 + \frac{ZY \tanh \frac{\theta}{2} \sinh \theta}{\theta^2}$$

$$= 1 + \frac{\cosh \theta - 1}{\sinh \theta} \sinh \theta = \cosh \theta$$

$$Z'\left(1 + \frac{Z'Y'}{4}\right) = \frac{2Z \tanh \frac{\theta}{2}}{\theta}\left(1 + \frac{ZY \tanh \frac{\theta}{2} \sinh \theta}{2\theta^2}\right)$$

$$= 2Z_0 \frac{\cosh \theta - 1}{\sinh \theta}\left(1 + \frac{\cosh \theta - 1}{2}\right) = Z_0 \sinh \theta$$

$$Y' = \frac{Y \sinh \theta}{\theta} = \frac{\sinh \theta}{Z_0}$$

The two sets of equations for terminal voltages and currents are, therefore, identical.

Next it will be demonstrated that the correct sending-end impedance is obtained, immaterial of loading, *i.e.*, when the line is free, grounded, and loaded by an impedance σ at the distant end.

Line Open:

$$Z_s' = \frac{Z'}{2} + \frac{1}{Y'} = \frac{Z \tanh \frac{\theta}{2}}{\theta} + \frac{\theta}{Y \sinh \theta}$$

$$= Z_0\left(\frac{\cosh \theta - 1}{\sinh \theta} + \frac{1}{\sinh \theta}\right) = Z_0 \coth \theta$$

which is correct for the open line.

Line Grounded:

$$Z_s' = \frac{Z'}{2} + \frac{\frac{Z'}{2} \times \frac{1}{Y'}}{\frac{Z'}{2} + \frac{1}{Y'}} = Z_0 \tanh \frac{\theta}{2} + \frac{Z_0 \tan \frac{\theta}{2} \times \frac{Z_0}{\sinh \theta}}{Z_0 \coth \theta}$$

$$= Z_0\left(\frac{\cosh \theta - 1}{\sinh \theta} + \frac{\cosh \theta - 1}{\sinh \theta \cosh \theta}\right) = Z_0 \tanh \theta$$

which is correct for the grounded line.

[1] See equations (83) and (101) used as conversion formulas in this proof.

Line Loaded by an Impedance σ:

$$Z_s' = \frac{Z'}{2} + \frac{\left(\frac{Z'}{2} + \sigma\right)\frac{1}{Y'}}{\frac{Z'}{2} + \sigma + \frac{1}{Y'}} = Z_0 \tanh\frac{\theta}{2} + \frac{\left(Z_0 \tanh\frac{\theta}{2} + \sigma\right)\frac{Z_0}{\sinh\theta}}{Z_0 \coth\theta + \sigma}$$

$$= \frac{Z_0}{\sinh\theta}\left[\cos\theta - 1 + \frac{\cosh\theta - 1 + \frac{\sigma}{Z_0}\sinh\theta}{\cosh\theta + \frac{\sigma}{Z_0}\sinh\theta}\right]$$

$$= Z_0\frac{\sinh\theta + \cosh\theta\tanh\theta'}{\cosh\theta + \sinh\theta\tanh\theta'}$$

$$= Z_0\frac{\tanh\theta + \tanh\theta'}{1 + \tanh\theta\tanh\theta'} = Z_0\tanh(\theta + \theta') = Z_0\tanh\delta_s$$

which is correct for the loaded line.

So far, it has been shown how an equivalent T or Π corresponding to a given smooth line can be obtained. Conversely, it is also possible to determine the constants of the conjugate smooth line which corresponds to a given T- or Π-circuit. The equations below give the necessary relations.

For T-circuit:

$$\theta = 2\sinh^{-1}\frac{\sqrt{Z'Y'}}{2} \tag{140}$$

$$Z_0 = \sqrt{\frac{Z'}{Y'}}\cosh\frac{\theta}{2} \tag{141}$$

For Π-circuit:

$$\theta = 2\sinh^{-1}\frac{\sqrt{Z''Y''}}{2} \tag{142}$$

$$Z_0 = \sqrt{\frac{Z''}{Y''}}\frac{1}{\cosh\frac{\theta}{2}} \tag{143}$$

Proof for T-line:[1]

$$Z'Y' = 2\tanh\frac{\theta}{2}\sinh\theta = \frac{4\tanh^2\frac{\theta}{2}}{1 - \tanh^2\frac{\theta}{2}} = 4\sinh^2\frac{\theta}{2}$$

[1] See equations (102), (103), and (104) used as conversion formulas in the proofs.

from which equation (140) immediately follows.

$$\frac{Z'}{Y'} = 2Z_0^2 \frac{\tanh \frac{\theta}{2}}{\sinh \theta} = Z_0^2\left(1 - \tanh^2 \frac{\theta}{2}\right) = \frac{Z_0^2}{\cosh^2 \frac{\theta}{2}}$$

which, solved for Z_0, gives equation (141).

Proof for Π *Line:*

$$Z''Y'' = 2 \tanh \frac{\theta}{2} \sinh \theta \text{ (as in } T \text{ case)}$$

Hence, equation (142) follows.

$$\frac{Z''}{Y''} = \frac{Z_0^2 \sinh \theta}{2 \tanh \frac{\theta}{2}} = \frac{Z_0^2}{1 - \tanh^2 \frac{\theta}{2}} = Z_0^2 \cosh^2 \frac{\theta}{2}$$

which, solved for Z_0, gives equation (143).

EXAMPLE 5

This example serves to demonstrate the numerical determination of the equivalent π-circuit of a long transmission line.

Statement of Problem

A three-phase, 200-mile, 150-kv., 60-cycle transmission line has the following constants per wire mile:

$$r = 0.158 \text{ ohm}$$
$$g = 0$$
$$L = 0.00212 \text{ henry}$$
$$C = 0.0141 \times 10^{-6} \text{ farad}$$

Determine the constants of the equivalent Π-circuit of this line.

Solution

Constants per mile

$$z = r + j\omega L = 0.158 + j377 \times 0.00212$$
$$= 0.158 + j0.799 = 0.814\underline{/78^\circ.8} \text{ ohm}$$
$$y = g + j\omega C = j377 \times 0.0141 \times 10^{-6} = j5.315 \times 10^{-6}$$
$$= 5.315 \times 10^{-6}\underline{/90^\circ} \text{ mho}$$
$$\alpha = \sqrt{zy} = 10^{-3}\sqrt{0.814\underline{/78^\circ.8} \times 5.315\underline{/90^\circ}} = 2.08 \times 10^{-3}\underline{/84^\circ.4} \text{ hyp}$$

Constants of entire line

$$Z = 200 \times 0.814\underline{/78^\circ.8} = 162.8\underline{/78^\circ.8} \text{ ohms}$$
$$Y = 200 \times 5.315 \times 10^{-6}\underline{/90^\circ} = 1.063 \times 10^{-3}\underline{/90^\circ} \text{ mho}$$
$$\theta = 200 \times 2.08 \times 10^{-3}\underline{/84^\circ.4} = 0.416\underline{/84^\circ.4} \text{ hyp}$$

Constants of equivalent Π

$$Z'' = Z\frac{\sinh\theta}{\theta} = 162.8\underline{/78^\circ.8}\frac{0.404\underline{/84^\circ.7}}{0.416\underline{/84^\circ.4}} = 158.0\underline{/79^\circ.1}$$

$$= 29.9 + j155.1 \text{ ohms}$$

$$\frac{Y''}{2} = Y\frac{\tanh\frac{\theta}{2}}{\theta} = 1.063 \times 10^{-3}\underline{/90^\circ}\frac{0.210\underline{/84^\circ.2}}{0.416\underline{/84^\circ.4}}$$

$$= 0.537 \times 10^{-3}\underline{/89^\circ.2} = (0.00752 + j0.537)10^{-3} \text{ mho}$$

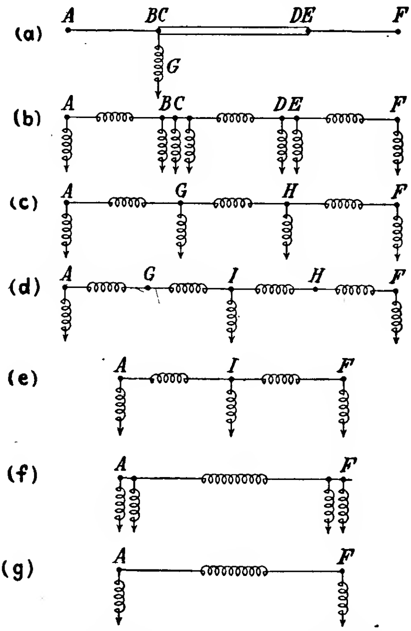

FIG. 110.—Equivalent Π representation of a composite line. The diagrams show the various steps necessary to obtain the equivalent Π by repeated applications of Y-Δ and Δ-Y transformations.

Equivalent Representation of Composite Lines.—A composite line can always be replaced by a single equivalent T- or Π-circuit which will correctly represent it as far as conditions at the terminals are concerned and at one definite frequency. These limitations are the same as were previously imposed on the equivalent T and Π representing a simple line. While a simple line always gave rise to symmetrical T's and Π's, however, the equivalent T- and Π-circuits of composite lines will, *in general, be dissymmetrical.* That is to say, the impedances of the arms of the T and the admittances of the pillars of the Π will be unequal.

One method of obtaining either the dissymmetrical T or Π of a composite line suggests itself immediately. Each section of the composite line may be replaced by an equivalent T or Π and the composite circuit so made up reduced to a single *T* or Π by repeated application of Y-Δ and Δ-Y transformations. Evidently the effect of casual impedance loads can also readily be taken into account by this method.

For instance, consider the network shown in Fig. 110. It consists of three simple line sections and an impedance load at the junction *BC*. Let it be desired to determine the equivalent Π. The reduction may be carried out as follows: Replace each section by an equivalent Π as in (*b*). Combine the pillar admittances at *DE* and the pillar admittances and the load σ at *BC*. The network shown in (*c*) results. The central Π-circuit *GH* may

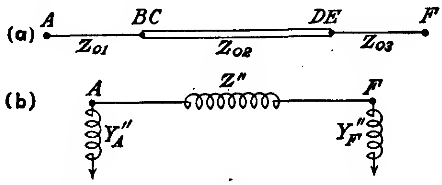

FIG. 111.—Three-section composite line.

now be changed into a T [*GIH* in (*d*)] by a Δ-Y transformation. The impedances *AG* and *GI*, and *IH* and *HF* are added, giving the circuit shown in (*e*). Converting the T-circuit *AIF* into a Π by a Y-Δ transformation gives (*f*). Finally, by paralleling the terminal leaks, the resultant equivalent Π as shown in (*g*) is obtained.

An alternative method[1] is also available for obtaining the equivalent T and Π of a composite line. This is convenient when the composite line consists of a series of simple sections without any intermediate loads. This method, however, will not be discussed in detail. Let it suffice to illustrate it by quoting the necessary formulas for the conversion of a three-section composite line [Fig. 111] to an equivalent Π.

The architrave impedance is given by

$$Z'' = Z_{01} \sinh \delta_A \cdot \frac{\cosh \delta_C}{\cosh \delta_B} \cdot \frac{\cosh \delta_E}{\cosh \delta_D} \tag{144}$$

$$= Z_{03} \sinh \delta_F \cdot \frac{\cosh \delta_D}{\cosh \delta_E} \cdot \frac{\cosh \delta_B}{\cosh \delta_C} \tag{145}$$

[1] KENNELLY, A. E., "Artificial Electric Lines," *loc. cit.*

The leak admittances are obtained from

$$Y''_A = \frac{1}{Z_{01} \tanh \delta_A} - \frac{1}{Z''} \tag{146}$$

$$Y''_F = \frac{1}{Z_{03} \tanh \delta_F} - \frac{1}{Z''} \tag{147}$$

Z_{01}, Z_{02}, etc. represent the surge impedances of the various sections. The δ's represent position angles at the point denoted by the subscripts with the line grounded at F in equations (144) and (146) and at A in equations (145) and (147).

Similar formulas can be established for reduction to the equivalent T.

Tests for the Determination of Line Constants.—If the sending-end impedance of a line is measured with the distant end of the line both grounded and free, the surge impedance Z_0 and hyperbolic angle θ can readily be calculated. The actual measurements may be made by various methods, for instance by an impedance bridge or by use of the *three-voltmeter* scheme.[1]

The sending-end impedances, grounded and free, are given by

$$Z_g = Z_0 \tanh \theta \tag{148}$$

$$Z_f = Z_0 \coth \theta \tag{149}$$

which, solved for Z_0 and θ, give

$$Z_0 = \sqrt{Z_g Z_f} \tag{150}$$

$$\theta = \tanh^{-1} \frac{Z_g}{Z_f} \tag{151}$$

Knowing the surge impedance and the hyperbolic angle, the total impedance and admittance can be found

$$Z = lz = l(r + j\omega L) = Z_0\theta \tag{152}$$

$$Y = ly = l(g + j\omega C) = \frac{\theta}{Z_0} \tag{153}$$

Evidently, if the length of the line is also known, the impedance and admittance per unit length can be determined, and from these the resistance, inductance, leakance, and capacitance per unit length. It is thus possible to obtain the fundamental constants by two simple tests.

[1] KENNELLY, A. E., "Artificial Electric Lines," *loc. cit.*

CHAPTER VIII

HARMONICS

It is desirable to operate alternating-current power circuits with voltages and currents of pure sinusoidal wave shapes. Distorted waves cause increased losses in lines and machines and hence a reduction in the efficiency of transmission and the efficiency of the connected machinery. Furthermore the presence of harmonics in rotating machines and transformers may result in other undesirable features, such as pulsating torque, vibrations, and overvoltages, due to resonance.

In present-day commercial practice, an almost ideal wave shape is, as a rule, realized under normal operating conditions when well-designed generating equipment and proper transformer connections are used. The harmonics which may be present are usually too small to be of importance as far as the power circuit is concerned. Although small, however, they may have considerable effect as regards inductive interference with neighboring communication lines.

The main sources of harmonics are:

1. The generators.
2. The transformers.
3. Corona.[1]

The Behavior of Harmonics on a Line.—Assume that a non-sinusoidal voltage is impressed on a transmission line and that it is desired to determine the distribution of voltage, current, and power on the line. It is not at all difficult to obtain the steady-state solution of this problem. The impressed distorted voltage is simply resolved into its fundamental and its higher harmonic

[1] Bennett, E., "An Oscillographic Study of Corona," *Trans. A.I.E.E.*, p. 1787, 1913.

Peek, F. W., Jr., "Voltage and Current Harmonics Caused by Corona," *Trans. A.I.E.E.*, p. 1155, 1921.

Whitehead, J. B., and N. Inouye, "Wave Form and Amplification of Corona Discharge," *Trans. A.I.E.E.*, p. 138, 1922.

Gardner, M. F., "Corona Investigation on an Artificial Line," *Trans. A.I.E.E.*, p. 897, 1925.

components. Each harmonic is then in itself a sinusoidal voltage and is treated as such. That is to say, the voltage, current, and power distribution of each harmonic frequency is computed separately by making use of the methods and equations developed in Chap. VI or VII for sinusoidal voltages. Having obtained the separate solutions, these are combined according to the well-known rules for the combination of quantities of different frequencies.

In carrying through such calculations, the proper constants must be used in connection with the various harmonics. In polyphase systems this also involves the possibility of different circuits existing for different groups of harmonics. Thus, in a three-phase system, the triple-frequency group (*i.e.*, the third harmonic and its multiples) and the group consisting of the other odd harmonics (*i.e.*, the fifth, seventh, eleventh, etc.) follow different paths. The harmonics in the former group appear as residuals, being in phase in the three phases. These currents, therefore, cannot flow unless there is a metallic neutral or a ground circuit for their return. The harmonics of the other group, being 120 deg. out of phase in the three phases, follow the same circuit as the fundamental.

When the circuit configuration is known, the constants for the harmonics of the second group are, as a rule, readily calculated (or selected from tables). The constants for the triple-frequency group, on the other hand, cannot always be obtained with precision. If a metallic neutral exists, exact calculations can usually be carried through. If there is a ground return, however, assumptions as to the position of the ground circuit or "equivalent ground plane" have to be made. The position of the latter depends on the condition of the soil, particularly on its moisture content, and may hence vary along the line. The equivalent ground plane may be located above the ground when this is covered with snow and as far down as 400 to 500 ft. below the surface in very dry regions.

Evidently, the position of the equivalent ground plane greatly affects the inductance and capacitance of the return circuit. The resistance of a ground return is very small and is usually neglected, even when the ground is dry and its resistivity high. This is due to the fact that the ground current distributes itself over so large an area, that the ground return circuit acts as a conductor of an immense cross-section.

The approximate order of magnitude of the resistance of a ground circuit may be estimated by making use of the formula for resistance between two hemispheres located, as shown in Fig. 112, at the surface of a homogeneous conducting medium extending to infinity in all directions. If the resistivity of the medium is ρ, the radius of the two hemispheres r_1 and r_2, respectively, and the distance between them d, the resistance of the circuit through which a current would flow from one hemisphere to the other is given by

$$R = \frac{\rho}{2\pi}\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{d}\right) \tag{1}$$

In applying this formula to a practical case, the ground plates are considered replaced by equivalent hemispheres of the same surface area as the plates and an estimated average value of soil resistivity is used.

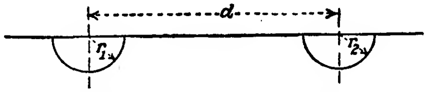

FIG. 112.—Two metallic hemispheres located at the surface of a homogeneous conducting medium extending to infinity in all directions.

In order to show that the resistance of the ground as a rule is small compared with the resistance of the outgoing metallic circuit and particularly compared with the reactance of the loop formed by the metallic conductor (or conductors) and the ground return, it may be well to consider a specific case. Assume a single No. 0000 copper conductor, 100 miles long with ground return. The area of the ground plates is 40 sq. ft. and the resistivity of the ground is assumed to be 500 ohm-cm. It is further assumed that the equivalent ground plane is located 30 ft. below the conductor.

The radius of the equivalent hemisphere is then

$$r = \sqrt{\frac{A}{2\pi}} = 12 \times 2.54\sqrt{\frac{40}{2\pi}} = 76.8 \text{ cm.}$$

and using equation (1) the resistance of the ground circuit becomes

$$R = \frac{500}{2\pi}\left[\frac{2}{76.8} - \frac{2}{100 \times 5{,}280 \times 12 \times 2.54}\right] = 2.07 \text{ ohms}$$

The resistance of 100 miles of No. 0000 copper conductor of 100 per cent conductivity is 25.9 ohms at 20°C. The ground

resistance, therefore, is in this case about 8 per cent of the resistance of the metallic conductor.

The inductance of the loop formed by the conductor and the ground return is

$$L = \left(741 \log_{10} \frac{2h}{r} + 80.5\right) 10^{-6}$$

$$= \left(741 \log_{10} \frac{60 \times 12}{0.23} + 80.5\right) 10^{-6}$$

$$= 2{,}673 \times 10^{-6} \text{ henry per mile}$$

and, hence, the total 60-cycle reactance

$$X = 100 \times 377 \times 2{,}673 \times 10^{-6} = 100.8 \text{ ohms}$$

The ground resistance is, as seen, only about 2 per cent of the reactance of the loop. Obviously, the impedance of the loop is affected only to a negligible extent by consideration of the ground resistance. Since the actual value of the latter is very uncertain, it is usually, and appropriately, ignored.

The complete voltage, current, and power distribution on a long line due to a high-frequency harmonic is illustrated in Example 1, Chap. VII. The calculations are carried through for a tooth-harmonic of twenty-nine times the fundamental frequency. This harmonic does not belong to the triple-frequency group and, hence, follows the same circuit as the fundamental.

EXAMPLE 1

This example shows how a harmonic of the triple-frequency group may be taken care of when present in a feeder circuit with metallic neutral. The circuit in this case is so short that only resistance and inductance need be taken into account. Calculation of the proper constants is included.

Statement of Problem

A short three-phase, 60-cycle feeder circuit with neutral is arranged as shown in Fig. 113.

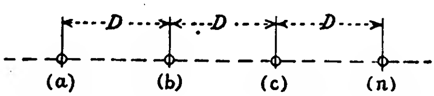

FIG. 113.—Configuration of three-phase feeder circuit with neutral.

The conductors a, b, and c are completely transposed with respect to each other, while the neutral n maintains its position on the right.

The spacing D is 2.5 ft., and the copper conductors are No. 0000 A.W.G. (diameter = 0.460 in., resistance = 0.259 ohm per mile). The length of the feeder is 2 miles; the load is Y-connected and balanced.

The voltmeters at the Y-connected generator show balanced line voltages of 13,200 volts and balanced voltages to neutral of 7,700 volts. The line ammeters show balanced currents of 350 amp. and the ammeter in the neutral indicates 60 amp. The polyphase wattmeter (current coils in line a and b and potential coils between a and c and between b and c) shows 6,802 kw., while the single-phase wattmeter (current coil in the neutral n and potential coil between c and n) indicates 39.6 kw. It is known that the *only harmonic present is the third.*

Compute the value of the phase and line voltages and the total power at the load.

Solution

Since the load is balanced and the conductors a, b, and c are completely transposed, the neutral will carry no current of fundamental frequency. The third-harmonic current will divide equally between a, b, and c and return on the neutral n. Hence, the system is balanced both as regards fundamental and third-harmonic quantities.

$$R = 2 \times 0.259 = 0.518 \text{ ohm per conductor}$$

Average fundamental reactance per conductor

$$\begin{aligned} X' &= 2\omega\left(741 \log_{10} \frac{\sqrt[3]{D_{ab} \times D_{bc} \times D_{ca}}}{r} + 80.50\mu\right)10^{-6} \\ &= 2\omega\left(741 \log_{10} \frac{D\sqrt[3]{2}}{r} + 80.5\mu\right)10^{-6} \\ &= 2 \times 377\left(741 \log_{10} \frac{1.26 \times 30}{0.23} + 80.5\right)10^{-6} \\ &= 1.300 \text{ ohms} \end{aligned}$$

The average number of third-harmonic flux linkages about any one of the three conductors (a), (b), and (c) per centimeter becomes

$$\lambda''' = \frac{I_n'''}{3}\left[2 \log \frac{1}{r} + \frac{\mu}{2} + \frac{1}{3}\left(2 \log \frac{1}{D_{ab}} + 2 \log \frac{1}{D_{ac}} + 2 \log \frac{1}{D_{bc}} + 2 \log \frac{1}{D_{ca}} + 2 \log \frac{1}{D_{ab}} + 2 \log \frac{1}{D_{bc}}\right) - 2 \log \frac{1}{D_{na}} - 2 \log \frac{1}{D_{nb}} - 2 \log \frac{1}{D_{nc}}\right]$$

$$= \frac{I_n'''}{3}\left[2 \log \frac{D_{na} \times D_{nb} \times D_{nc}}{r\sqrt[3]{(D_{ab} \times D_{bc} \times D_{ca})^2}} + \frac{\mu}{2}\right]$$

$$= \frac{I_n'''}{3}\left(2 \log \frac{D\sqrt[3]{54}}{r} + \frac{\mu}{2}\right)$$

Average third-harmonic reactance per conductor

$$\begin{aligned} X''' &= 6\omega\left(741 \log_{10} \frac{D\sqrt[3]{54}}{r} + 80.5\right)10^{-6} \\ &= 6 \times 377\left(741 \log_{10} \frac{3.78 \times 30}{0.23} + 80.5\right)10^{-6} = 4.696 \text{ ohms} \end{aligned}$$

The average number of third-harmonic flux linkages about the neutral (n) per centimeter becomes

$$\lambda_n''' = I_n'''\left[2\log\frac{1}{r} + \frac{\mu}{2} - \frac{1}{3}\left(2\log\frac{1}{D_{na}} + 2\log\frac{1}{D_{nb}} + 2\log\frac{1}{D_{nc}}\right)\right.$$
$$= I_n'''\left[2\log\frac{\sqrt[3]{D_{na}\times D_{nb}\times D_{nc}}}{r} + \frac{\mu}{2}\right]$$
$$= I_n'''\left[2\log\frac{D\sqrt[3]{6}}{r} + \frac{\mu}{2}\right]$$

Average third-harmonic reactance of the neutral

$$X_n''' = 6\omega\left(741\log_{10}\frac{D\sqrt[3]{6}}{r} + 80.5\right)10^{-6}$$
$$= 6\times 377\left(741\log_{10}\frac{1.817\times 30}{0.23} + 80.5\right)10^{-6} = 4.163\text{ ohms}$$

Fundamental phase voltage at the generator

$$V_g' = \frac{13{,}200}{\sqrt{3}} = 7{,}622\text{ volts}$$

Third-harmonic phase voltage at the generator

$$V_g''' = \sqrt{7{,}700^2 - \frac{13{,}200^2}{3}} = 1{,}100\text{ volts}$$

Fundamental line currents

$$I' = \sqrt{350^2 - 20^2} = 349.4\text{ amp.}$$

Fundamental power factor at the generator

$$\cos\theta' = \frac{6{,}802}{\sqrt{3}\times 13.2\times 349.4} = 0.8515$$

Third-harmonic power factor at the generator

$$\cos\theta''' = \frac{39{,}600}{1{,}100\times 60} = 0.6000$$

Fundamental phase voltage at the load

$$V_{\text{load}}' = V_g' - I'(R + jX')$$
$$= 7{,}622 - 349.4(0.8515 - j0.5243)(0.518 + j1.300)$$
$$= 7{,}230 - j292 = 7{,}236\text{ volts}$$

Third-harmonic phase voltage at the load

$$V_{\text{load}}''' = V_g''' - \frac{I_n'''}{3}[4R + j(X''' + 3X_n''')]$$
$$= 1{,}100 - 20(0.6 - j0.8)(2.072 + j17.185)$$
$$= 800 - j173 = 818.5\text{ volts}$$

Line voltages at the load

$$V_{\text{line}} = \sqrt{3}\times 7{,}236 = 12{,}533\text{ volts}$$

Phase voltages at the load

$$V_{\text{load}} = \sqrt{7{,}236^2 + 818.5^2} = 7{,}282 \text{ volts}$$

Fundamental power per phase at the load

$$P'_{\text{load}} = (7{,}230 \times 0.8515 + 292 \times 0.5243)349.4 \times 10^{-3}$$
$$= 2{,}204.5 \text{ kw.}$$

Third-harmonic power per phase at the load

$$P'''_{\text{load}} = (800 \times 0.6 + 173 \times 0.8)20 \times 10^{-3} = 12.368 \text{ kw.}$$

Total power at the load

$$P_o = 3(2{,}204.5 + 12.37) = 6{,}650.6 \text{ kw.}$$

Generator Harmonics.—The harmonics which appear in the open-circuit line voltage of a well-designed generator are usually so small as to be negligible. The line voltages of such machines, therefore, may as a rule be considered strictly sinusoidal.

By properly designing the shape of the pole shoes of a salient-pole machine and by properly arranging and distributing the field windings of a non-salient-pole machine, the distribution of air-gap flux due to the field is made very nearly sinusoidal. It may be impracticable to get a *perfect* flux distribution, but the harmonic components can always be made *very small.* These harmonic fluxes, if present, will induce harmonic voltages in the armature windings. It is well known, however, that by properly distributing the latter, the values of these harmonic voltages may be made *extremely small.* By making use of this property in conjunction with the fact that no harmonic of the triple-frequency group (*i.e.*, third, ninth, fifteenth, etc.) when generated by fluxes, stationary with respect to the poles, and having space distributions in the air gap corresponding to frequencies in this group, can appear between lines of a three-phase system, there is obviously excellent possibility of ridding the open-circuit line voltages of all the important harmonics.[1]

When a non-salient-pole machine carries a balanced load, the armature reaction will not distort the flux and the voltages will remain sinusoidal. In a salient-pole machine, on the other hand, the armature reaction will distort the air-gap flux to a certain extent, except at power factors in the neighborhood of zero. Some of the harmonics introduced in the flux will be reflected in

[1] The harmonics in the air-gap flux and in the induced voltage of a synchronous generator can be predicted with good accuracy from design data. A general method applicable to non-salient-pole machines is given by W. V. Lyon in "Application of Harmonic Analysis to the Theory of Synchronous Machines," *Trans. A.I.E.E.*, p. 1122, 1918.

the armature voltages unless suppressed by the connection or distribution of the armature windings. There is a chance, therefore, that some harmonics may appear in the line voltages of a salient-pole machine under balanced load. These harmonics, however, should always be rather insignificant.

When a synchronous machine carries an unbalanced load, harmonics in the phase and line voltages are unavoidable. This will be the case whether the generator is a salient- or a non-salient-pole machine. These harmonics may, in many instances, reach appreciable magnitudes. In order to get an idea of how these harmonics originate, it is convenient to consider the unbalanced armature currents equivalent to a positive- and a negative-sequence balanced system of currents. It is assumed that the generator has no neutral connection so that no zero-sequence currents can flow. The armature reaction due to the positive-sequence currents is stationary with respect to the field poles and produces no flux distortion and hence no harmonic voltages except as discussed above for the balanced load on the salient-pole machine.

The armature reaction due to the negative-sequence currents sweeps by the poles at twice synchronous speed and induces by transformer action second-harmonic currents in the field windings, second-harmonic eddy currents in the iron, and also second-harmonic currents in the damper windings, if such windings are present. The field windings represent single-phase windings and the currents induced in them are, therefore, strictly single-phase. The eddy currents induced in the iron and the currents in the dampers are of a three-phase or semi-three-phase nature.

The pulsating second-harmonic magnetomotive force of the field is equivalent to two fields revolving in the air gap in opposite directions with twice synchronous speed with respect to the field poles. Obviously, one of these fields rotates at synchronous speed with respect to the armature and is stationary with respect to the negative-sequence armature reaction, while the other rotates with three times synchronous speed with respect to the armature. The former tends to compensate for a part of the negative-sequence reaction, the latter induces *third-harmonic voltages* in the armature windings. These third-harmonic voltages are caused by fluxes of fundamental space distribution. Hence, they will be 120 deg. phase-displaced in the three phases and will *appear in the terminal voltages.*

The second-harmonic eddy currents and damper currents of true three-phase nature produce only revolving magnetomotive forces stationary with respect to the negative-sequence reaction and will contribute to the compensation of the latter. If the eddy currents and damper currents are also to some extent of a single-phase nature, the single-phase parts will have an effect similar to that of the field windings proper. Hence they will produce oppositely revolving fields in the air gap. They will, therefore, contribute to the compensation of the negative-sequence armature reaction and also induce third-harmonic voltages in the armature windings. These third-harmonic voltages will appear between lines as did the third-harmonic voltage caused by the second-harmonic currents in the field windings.

If the unbalanced load offers infinite impedance to third-harmonic currents, no currents of this frequency can flow. In such cases a triple-frequency voltage is the only harmonic voltage introduced by the unbalance. If, on the other hand, there is an opportunity for unbalanced third-harmonic currents to flow, a fifth-harmonic component may be introduced in the phase and line voltages.

This is readily seen by resolving the third-harmonic currents into a positive- and a negative-sequence system. The positive-sequence reaction revolves at twice synchronous speed, the negative-sequence reaction with four times synchronous speed with respect to the field poles. The former induces second-harmonic currents, the latter fourth-harmonic currents in the field windings, the iron, and the dampers. The second-harmonic currents in the field windings and the second-harmonic currents of single-phase nature in the iron and in the dampers will react back on the armature at fundamental and triple frequency. The fourth-harmonic field currents and the other fourth-harmonic single-phase currents will react back on the armature at third-harmonic and fifth-harmonic frequency. A fifth-harmonic voltage is thus introduced.

Similarly, if fifth-harmonic unbalanced currents flow, a seventh-harmonic voltage may appear. In general, therefore, when an unbalanced harmonic current flows in the armature circuit, a harmonic voltage of the next higher order will be introduced in the voltage. The magnitude of these harmonic voltages, however, rapidly decreases at the order goes up.

Transformer Harmonics.—It was brought out in the preceding discussion that the harmonics in the generator voltages, as a rule, were negligibly small under ordinary operating conditions. They could be taken care of and eliminated by proper design. The harmonics due to transformers, on the other hand, are caused by definite and inherent properties of iron and steel which cannot be changed by human control.

The harmonics can be decreased only by operating at reduced saturation. This is not done, however, on account of the less economical design which it would entail. Quite to the contrary, the trend in transformer design has been a distinct and continuous increase in operating densities with a corresponding increase in voltage or current harmonics.

The transformer harmonics, therefore, cannot be eliminated at the source. It is fortunate, however, that in a three-phase system the distribution of the harmonics between primary and secondary circuits can be partially controlled by the connections used.

The distribution of harmonics in single-phase and three-phase transformer banks is discussed below in detail.[1]

Single-phase Transformers.—It is a well-known fact that, when a sinusoidal voltage is impressed upon a single-phase transformer, the exciting current that it takes will be non-sinusoidal. The number and magnitude of the harmonics which the exciting current contains depend upon the characteristics of the iron and the maximum density at which it is operated.

The third harmonic is generally by far the most prominent, while usually an appreciable fifth also is present. Higher odd harmonics are also easily traced, but their magnitude is, in general, very small. These higher harmonics are, therefore, of minor importance as far as the operation of the transformer is concerned. At normal saturation, the fundamental is about 90 per cent, the

[1] FRANK, J. J., "Observation of Harmonics in Current and in Voltage Wave Shapes of Transformers," *Trans. A.I.E.E.*, p. 809, 1910.

PETERS, J. F., "Harmonics in Transformer Magnetizing Currents," *Trans. A.I.E.E.*, p. 557, 1915.

LYON, W. V., "Obtaining Approximate Values of Harmonics," *Elec. World*, p. 949, 1917.

FACCIOLI, G., "Triple Harmonics in Transformers," *Jour. A.I.E.E.*, p. 351, 1922.

DAHL, O. G. C., "Transformer Harmonics and Their Distribution," *Trans. A.I.E.E.*, p. 351, 1925.

third harmonic about 45 per cent, and the fifth harmonic about 15 per cent of the equivalent sinusoidal exciting current. The percentage harmonics increase with the flux density as long as

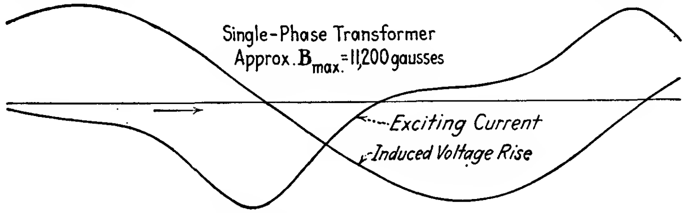

FIG. 114.—Oscillogram of the exciting current of a transformer with sinusoidal voltage impressed.

this is not forced up to abnormal values. At such abnormal saturations, the percentage harmonics may be expected to decrease.

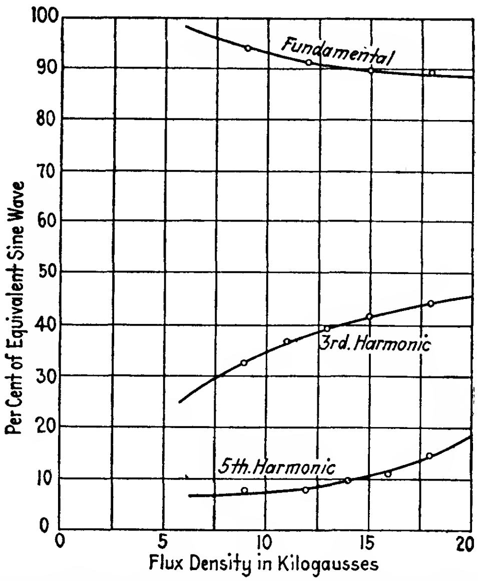

FIG. 115.—Harmonic components in the exciting current of a transformer with sinusoidal voltage impressed. These curves show the variation in the harmonics with saturation.

Figure 114 shows an oscillogram of single-phase exciting current. The magnitudes of the fundamental, third-harmonic, and fifth-harmonic components, as found from analysis of a series of oscillograms, are given by the curves of Fig. 115. The curves are

plotted against fundamental flux density and it will be noted that the percentage harmonics increases with the density.

As a typical example, the results of the analysis of the exciting-current curve in the oscillogram Fig. 114 are plotted in Fig. 116. The relative magnitudes and phase relations are correctly reproduced. The phase relation between each harmonic current, for

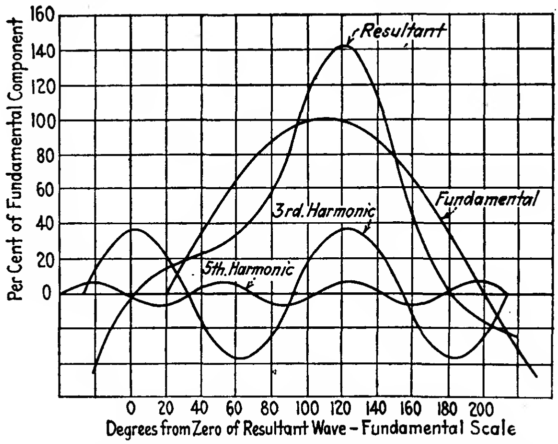

FIG. 116.—Components of exciting current from analysis of the oscillogram in Fig. 114. The relative magnitudes as well as phase relationships of the harmonics are correctly reproduced.

instance, the third, and the corresponding third-harmonic electromotive force depends on the triple-frequency impedance of the circuits where it flows as established by equations (6) and (7).

In the discussion of distribution of harmonics which follows, the third harmonic alone is mentioned and the equations have been established for this harmonic. It should be noted, however, that

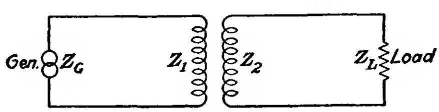

FIG. 117.—Two-circuit transformer with impedance load on the secondary.

in single-phase connections all harmonics follow the same laws and what is said in regard to the third harmonic will hold for any harmonic.

Consider the third-harmonic components in the transformer circuit shown in Fig. 117. In general, third-harmonic currents will flow in the secondary as well as in the primary circuit.

The induced voltage of the generator is assumed to be strictly sinusoidal. Since third-harmonic currents exist, they must be produced by triple-frequency electromotive forces. Being sinusoidal, the generator voltage cannot directly give rise to third-harmonic currents. Hence, the triple-frequency electromotive forces are generated by a triple-frequency flux in the iron core. Assuming unity ratio of transformation, or referring all quantities to the same side, the third-harmonic electromotive forces induced in the two windings by this flux are equal in magnitude and phase. The following equations[1] interrelate the third-harmonic quantities:

$$V_1''' = I_1'''Z_G''' = E_{1c}''' - I_1'''Z_{11}''' - jX_{12}'''I_2''' \tag{2}$$
$$V_2''' = I_2'''Z_L''' = E_{2c}''' - I_2'''Z_{22}''' - jX_{21}'''I_1''' \tag{3}$$
$$Z_{11}'' = R_1 + j(X_1''' + X_{12}''') \tag{4}$$
$$Z_{22}''' = R_2 + j(X_2''' + X_{21}''') \tag{5}$$

The symbols used in these equations have the following meanings:

$E_{1c}''' = E_{2c}'''$ = third-harmonic electromotive forces induced in the windings by the flux in the core
V_1''', V_2''' = third-harmonic terminal voltages
I_1''', I_2''' = third-harmonic currents
Z_{11}''', Z_{22}''' = third-harmonic self impedances
Z_1''', Z_2''' = third-harmonic leakage impedances
X_{12}''', X_{21}''' = third-harmonic mutual reactances
Z_G''' = third-harmonic impedance of generator and primary leads
Z_L''' = third-harmonic impedance of load and secondary leads

It is apparent that it is impossible to get entirely rid of a small third-harmonic component in the terminal voltages of a transformer, even if the generator voltage is a pure sine wave. The complete elimination of the third-harmonic voltage components would require zero triple-frequency impedance of the circuits (or

[1] Note that in this discussion of transformer harmonics it has been considered convenient to let the symbols V of terminal voltage represent *voltage drops* and the symbol E_c of induced voltage represent a *voltage rise*, since the harmonic voltages are generated in the transformer itself. In the general transformer equations previously developed in Chap. II, the opposite significance was attached to these symbols.

one of the circuits) where the third-harmonic current flows, a condition which, of course, never can be fulfilled.

In single-phase or polyphase connections, however, where no appreciable impedance is offered to the flow of the third-harmonic current, the triple-frequency voltages will usually be so small as to be entirely negligible in comparison with the fundamental component. The resultant voltages, therefore, will be sensibly sinusoidal.

Not only the magnitude, but also the phase of the impedance which the third-harmonic current must overcome, has an important effect on the magnitude of the third-harmonic voltages and currents. Without going into the question in further detail, it may be said that a lagging third-harmonic current, in general, reduces the triple-frequency flux in the core, while a leading current tends to amplify it.

Transposing terms, equations (2) and (3) may be written

$$E_{1c}''' = I_1'''(Z_{11}''' + Z_G''') + jI_2'''X_{12}''' \tag{6}$$

$$E_{2c}''' = I_2'''(Z_{22}''' + Z_L''') + jI_1'''X_{21}''' \tag{7}$$

By equating these expressions the distribution of the third-harmonic current between the primary and secondary is obtained. The ratio of the currents is

$$\frac{I_1'''}{I_2'''} = \frac{Z_{22}''' + Z_L''' - jX_{12}'''}{Z_{11}''' + Z_G''' - jX_{21}'''} = \frac{Z_2''' + Z_L'''}{Z_1''' + Z_G'''} \tag{8}$$

Equation (8) shows that the distribution of third-harmonic current between the two windings depends upon the leakage impedance of the windings and upon the external impedances. The distribution is inversely proportional to the ratio of the total impedances of the two circuits.

Usually, the load impedance will be very much greater than the other impedances involved (the two leakage impedances and the generator impedance) when referred to the secondary side. Hence, in general, the third-harmonic current in the primary will be many times greater than in the secondary when both currents are referred to the same side. If they are not referred to the same side, the actual third-harmonic current in the primary may, of course, be the smaller in the case of a step-down transformer of large ratio.

It may be said, then, that the distribution of third-harmonic current between the two windings of a transformer operating

single-phase is largely regulated by the magnitude and character of the triple-frequency impedance of the load. The minimum fundamental impedance of the load is determined by the rating of the transformer. The load, however, may easily be made large to the fundamental and still extremely small to the third harmonic by a suitable combination of inductance and capacitance in series. Let

$$Z_2''' + Z_L''' = R_2 + jX_2''' + R_L + j(X_L''' - X_C''') \tag{9}$$

If, then,

$$X_2''' + X_L''' - X_C''' = 0 \tag{10}$$

the secondary is tuned to series resonance for the third-harmonic current. If, at the same time, the load resistance is zero, the ratio of the secondary third-harmonic current to the primary third-harmonic current is a maximum, the distribution being given by

$$\frac{I_1'''}{I_2'''} = \frac{R_2}{Z_1''' + Z_G'''} \tag{11}$$

It should not be inferred from this statement that the secondary "leakage-resonance" condition gives rise to the maximum amount of third-harmonic current. Much larger currents as well as voltages may be obtained at other capacitive loads and at a second resonance condition. Operation in this region is also at certain points accompanied by peculiar instability phenomena.[1]

There has been considerable discussion of the causes for the harmonics in the magnetizing current and the voltage of a transformer. While opinions on this question have differed a good deal in earlier years, most engineers now agree they are caused both by the varying permeability of the iron and by hysteresis.[2] The relation of the harmonics to the power losses in the core, however, has not, as far as the writer is aware, been settled to everybody's satisfaction. The writer's conception of this question is outlined in the following paragraphs.

If the voltage of the generator is strictly sinusoidal, then none of the harmonic currents can produce power in conjunction with this voltage. Hence, power is input to the core at fundamental frequency only.

[1] GOULD, KING E., "Instability in Transformer Banks," paper presented at the Regional Meeting of the *A.I.E.E.*, Pittsfield, Mass., May, 1927.

[2] Excellently discussed by J. J. FRANK, *loc. cit.*

The harmonic currents, however, will necessarily give rise to a copper loss in the circuits where they exist. The power corresponding to this copper loss, plus the losses in the core caused by the non-fundamental fluxes, is then evidently conveyed to the core as power of fundamental frequency. In the core it is converted to power of other frequencies, a part of which is given out to the circuits where the currents of the higher frequencies flow.

The transformer core is in this respect nothing but a frequency converter, and the harmonics add to what may be called the *apparent core loss,* or to the fundamental input to the core, while, in reality, a part of this power is expended as copper loss by harmonic currents.

Conceive a hypothetical transformer having core loss but requiring no harmonics in the magnetizing current for impressed sinusoidal voltage. The vector diagram on open circuit is given in

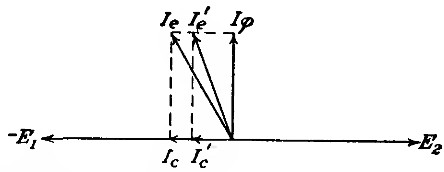

FIG. 118.—Vector diagram showing increase in apparent core loss.

Fig. 118, and the exciting current taken is I'_e. The hypothetical core is now exchanged for a regular iron core requiring harmonics in the magnetizing current. Figure 118 may now be used as a vector diagram of fundamental quantities only. Neglecting the slight change in the fundamental leakage-impedance drop of the primary, the only effect of the sudden introduction of harmonics in the current would be to increase the excitation conductance of the transformer. It would now draw a fundamental current I_e, and the apparent increase in core loss is

$$E_1(I_c - I'_c) \text{ watts} \tag{12}$$

This power is, as already pointed out, not utilized as power of fundamental frequency but is converted to power of higher frequencies, part of which is absorbed by the core (and this part may be very small if the non-fundamental fluxes are small), while the rest is dissipated as copper loss in the circuits carrying the harmonic currents.

Three-phase Connections of Single-phase Transformers.—When single-phase transformers are connected for three-phase operation, the method of connection constitutes a means by which the distribution of the third harmonics and multiples may be partly controlled independent of the external circuits. This depends upon the well-known fact that the triple-frequency voltages and currents in a balanced system are in phase in the three phases; in other words they appear as residuals.

The third-harmonic voltage, therefore, cannot appear between lines while it may be found as a component of the voltage to neutral. The third-harmonic current can appear on the lines only in a Y-connection with neutral. In a Δ-connection it will circulate in the closed delta but cannot escape from this and enter the lines.

The other harmonics which are not multiples of three, and hence are phase-displaced 120 deg. in the three phases, cannot be controlled by transformer connections independent of the external circuits.

In the following discussion of the distribution of the harmonics, strictly sinusoidal impressed voltages and balanced transformers are assumed. The ratio of transformation of the transformers is assumed to be unity.

Since the same laws do not govern the distribution of the two classes of harmonics, *i.e.*, the third harmonic and its multiples and those which are not multiples of three, equations for the fifth as well as for the third harmonics have been established. The generator and the balanced load are both assumed Y-connected for all transformer connections. The equations given, however, are easily modified to hold when either generator or load, or both, are delta-connected. *No load*, in all cases, means that the secondary lines are opened at the transformer terminals.

It should be noted that voltages and currents of triple frequency are given *per transformer*. In a Y-connection, this is equivalent to *voltage to neutral* and *line current*. Fifth-harmonic voltages and currents are *to neutral* and *per line*, respectively, for any connection used.

This significance of the symbols used in the equations below should be kept in mind, as otherwise the equations are easily open to incorrect interpretation.

(*a*) Δ-Δ *Connection.*—With this connection (Fig. 119) the third-harmonic currents cannot appear on the lines, but will exist as

circulating currents in the two deltas. Conditions are the same whether the transformer bank is loaded or not.

The induced third-harmonic electromotive forces are given by

$$E_{1c}''' = E_{2c}''' = I_1'''Z_{11}''' + jI_2'''X_{12}''' = I_2'''Z_{22}''' + jI_1'''X_{21}''' \quad (13)$$

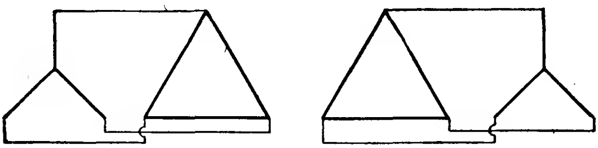

FIG. 119.

and the division of third-harmonic current between primary and secondary is given by

$$\frac{I_1'''}{I_2'''} = \frac{Z_2'''}{Z_1'''} \quad (14)$$

The fifth-harmonic currents will appear on the lines. When the transformers are loaded the following equations hold:

$$E_{1c}^V = E_{2c}^V = I_1^V\left(\frac{Z_{11}^V}{3} + Z_G^V\right) + jI_2^V\frac{X_{12}^V}{3}$$

$$= I_2^V\left(\frac{Z_{22}^V}{3} + Z_L^V\right) + jI_1^V\frac{X_{21}^V}{3} \quad (15)$$

$$\frac{I_1^V}{I_2^V} = \frac{Z_2^V + 3Z_L^V}{Z_1^V + 3Z_G^V} \quad (16)$$

When the load is disconnected (at the secondary terminals of the bank), the fifth-harmonic current ceases to flow in the secondary windings, while a fifth-harmonic voltage V_2^V still appears on the secondary side. The following relations hold:

$$E_{1c}^V = E_{2c}^V = I_1^V\left(\frac{Z_{11}^V}{3} + Z_G^V\right) \quad (17)$$

$$V_2^V = E_{2c}^V - jI_1^V\frac{X_{21}^V}{3} = I_1^V\left(\frac{Z_1^V}{3} + Z_G^V\right) \quad (18)$$

(*b*) *Δ-Y Connection. Isolated Neutrals.*—With this connection (Fig. 120), the third-harmonic current will be confined to the primary delta, both when the bank is loaded and when it is on open circuit.

The following equations will hold:

$$E_{1c}''' = E_{2c}''' = I_1'''Z_{11}''' \tag{19}$$

$$V_2''' = E_{2c}''' - jI_1'''X_{21}''' = I_1'''Z_1''' \tag{20}$$

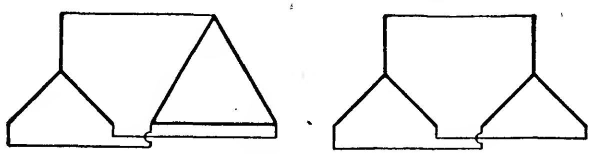

FIG. 120.

The equations for the fifth harmonic when the bank is loaded are

$$E_{1c}^V = \frac{E_{2c}^V}{\sqrt{3}}\overline{\backslash 30°} = I_1^V\left(\frac{Z_{11}^V}{3} + Z_G^V\right) + j\frac{I_2^V}{\sqrt{3}}X_{12}^V\overline{\backslash 30°}$$

$$= \frac{I_2^V}{\sqrt{3}}(Z_{22}^V + Z_L^V)\overline{\backslash 30°} + jI_1^V\frac{X_{21}^V}{3} \tag{21}$$

$$\frac{I_1^V}{I_2^V} = \frac{\sqrt{3}(Z_2^V + Z_L^V)}{Z_1^V + 3Z_G^V}\overline{\backslash 30°} \tag{22}$$

Disconnecting the load gives

$$E_{1c}^V = \frac{E_{2c}^V}{\sqrt{3}}\overline{\backslash 30°} = I_1^V\left(\frac{Z_{11}^V}{3} + Z_G^V\right) \tag{23}$$

$$V_2^V = E_{2c}^V - j\frac{I_1^V}{\sqrt{3}}X_{21}^V\underline{/30°} = \sqrt{3}I_1^V\left(\frac{Z_1^V}{3} + Z_G^V\right)\underline{/30°} \tag{24}$$

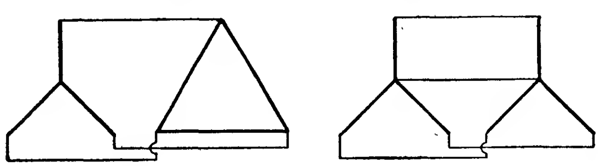

FIG. 121.

Interconnected Neutrals.—When the bank is loaded (Fig. 121), the following equations hold for the third harmonic

$$E_{1c}''' = E_{2c}''' = I_1'''Z_{11}''' + jI_2'''X_{12}'''$$

$$= I_2'''(Z_{22}''' + Z_L''' + 3Z_{2n}''') + jI_1'''X_{21}''' \tag{25}$$

$$\frac{I_1'''}{I_2'''} = \frac{Z_2''' + Z_L''' + 3Z_{2n}'''}{Z_1'''} \tag{26}$$

When the load is removed, equations (19) and (20) express the triple-frequency relations.

The fifth harmonic will follow the same laws as in the case with isolated neutral and is hence determined by equations (21), (22), (23), and (24).

(c) *Y-Δ Connection. Isolated Neutrals.*—With this connection (Fig. 122), the third-harmonic current will be exclusively confined

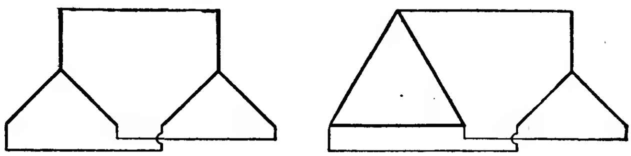

FIG. 122.

to the secondary delta both for load and no-load conditions. The following equations hold:

$$E_{1c}''' = E_{2c}''' = I_2'''Z_{22}''' \tag{27}$$

$$V_1''' = E_{1c}''' - jI_2'''X_{12}''' = I_2'''Z_2''' \tag{28}$$

When the bank is loaded the fifth-harmonic relations are

$$E_{1c}^{V} = \sqrt{3}\,E_{2c}^{V}\underline{/30^\circ} = I_1^{V}(Z_{11}^{V} + Z_G^{V}) + j\frac{I_2^{V}}{\sqrt{3}}X_{12}^{V}\underline{/30^\circ}$$

$$= \sqrt{3}\,I_2^{V}\left(\frac{Z_{22}^{V}}{3} + Z_L^{V}\right)\underline{/30^\circ} + jI_1^{V}X_{21}^{V} \tag{29}$$

$$\frac{I_1^{V}}{I_2^{V}} = \frac{Z_2^{V} + 3Z_L^{V}}{\sqrt{3}(Z_1^{V} + Z_G^{V})}\underline{/30^\circ} \tag{30}$$

Removing the load gives

$$E_{1c}^{V} = \sqrt{3}E_{2c}^{V}\underline{/30^\circ} = I_1^{V}(Z_{11}^{V} + Z_G^{V}) \tag{31}$$

$$V_2^{V} = E_{2G}^{V} - j\frac{I_1^{V}}{\sqrt{3}}X_{21}^{V}\overline{\backslash 30^\circ} = \frac{I_1^{V}}{\sqrt{3}}(Z_1^{V} + Z_G^{V})\overline{\backslash 30^\circ} \tag{32}$$

Interconnected Neutrals.—With this connection (Fig. 123), the third-harmonic current will flow in the primary lines and the

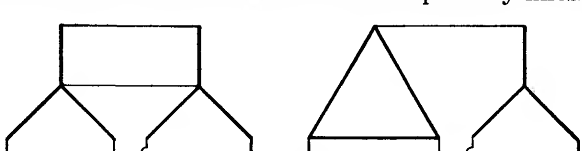

FIG. 123.

secondary delta both at no load and when the bank is loaded. The following equations hold:

$$E_{1c}''' = E_{2c}''' = I_1'''(Z_{11}''' + Z_G''' + 3Z_{1n}''') + jI_2'''X_{12}''' = I_2'''Z_{22}''' + jI_1'''X_{21}''' \quad (33)$$

$$\frac{I_1'''}{I_2'''} = \frac{Z_2'''}{Z_1''' + Z_G''' + 3Z_{1n}'''} \quad (34)$$

The neutral connection does not affect the fifth harmonics; they still follow equations (29), (30), (31), and (32).

(*d*) *Y-Y Connection. All Neutrals Isolated.*—With this connection (Fig. 124), no third-harmonic current can flow on either

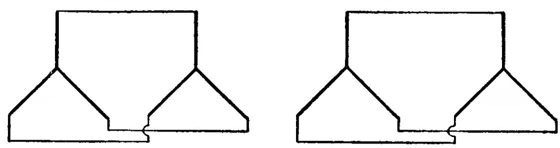

FIG. 124.

side under any condition. A third-harmonic voltage, however, which may be very large, exists on both sides.

$$V_1''' = V_2''' = E_{1c}''' = E_{2c}''' \quad (35)$$

When the bank is loaded, the equations for the fifth harmonic are

$$E_{1c}^{V} = E_{2c}^{V} = I_1^{V}(Z_{11}^{V} + Z_G^{V}) + jI_2^{V}X_{12}^{V} = I_2^{V}(Z_{22}^{V} + Z_L^{V}) + jI_1^{V}X_{21}^{V} \quad (36)$$

$$\frac{I_1^{V}}{I_2^{V}} = \frac{Z_2^{V} + Z_L^{V}}{Z_1^{V} + Z_G^{V}} \quad (37)$$

Disconnecting the load gives

$$E_{1c}^{V} = E_{2c}^{V} = I_1^{V}(Z_{11}^{V} + Z_G^{V}) \quad (38)$$

$$V_2^{V} = E_{2c}^{V} - jI_1^{V}X_2^{V} = I_1^{V}(Z_1^{V} + Z_G^{V}) \quad (39)$$

Primary and Generator Neutrals Interconnected.—With this connection (Fig. 125), the third-harmonic current will flow on the

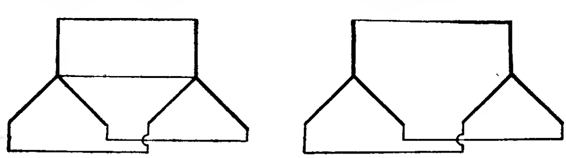

FIG. 125.

primary side, both at no load and when load is put on the bank. The equations for the third harmonic are

$$E_{1c}''' = E_{2c}''' = I_1'''(Z_{11}''' + Z_G''' + 3Z_{1n}''') \tag{40}$$

$$\begin{aligned} V_2''' &= E_{2c}''' - jI_1'''X_{21}''' \\ &= I_1'''(Z_1''' + Z_G''' + 3Z_{1n}''') \end{aligned} \tag{41}$$

The conditions as far as the fifth harmonics are concerned are identical with those existing when all neutrals are isolated, as expressed by equations (36), (37), (38), and (39).

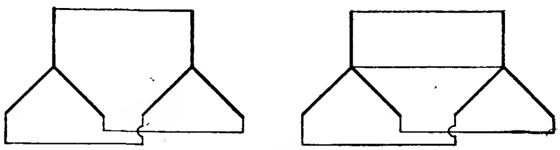

FIG. 126.

Secondary and Load Neutrals Interconnected.—When the bank is loaded (Fig. 126), the third-harmonic current will flow on the secondary side.

$$E_{1c}''' = E_{2c}''' = I_2'''(Z_{22}''' + Z_L''' + 3Z_{2n}''') \tag{42}$$

$$\begin{aligned} V_1''' &= E_{1c}''' - jI_2'''X_{12}''' \\ &= I_2'''(Z_2''' + Z_L''' + 3Z_{2n}''') \end{aligned} \tag{43}$$

At no load there is no circuit in which the third-harmonic current can flow. Equation (35) holds for the voltages.

The fifth harmonics behave as in the other Y-Y connections (equations (36), (37), (38), and (39)).

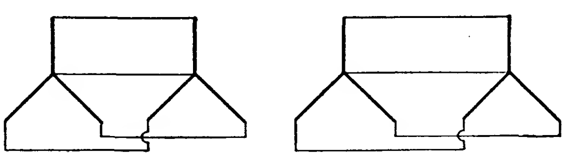

FIG. 127.

Neutrals Interconnected on Both Sides.—With this connection (Fig. 127), a third-harmonic current will flow on both sides when the bank is loaded.

$$\begin{aligned} E_{1c}''' = E_{2c}''' &= I_1'''(Z_{11}''' + Z_G''' + 3Z_{1n}''') + jI_2'''X_{12}''' \\ &= I_2'''(Z_{22}''' + Z_L''' + 3Z_{2n}''') + jI_1'''X_{21}''' \end{aligned} \tag{44}$$

$$\frac{I_1'''}{I_2'''} = \frac{Z_2''' + Z_L''' + 3Z_{2n}'''}{Z_1''' + Z_G''' + 3Z_{1n}'''} \tag{45}$$

When the load is removed, equations (40) and (41) will evidently hold. The fifth harmonics follow equations (36), (37), (38), and (39).

(e) *Y-Y Connection with Tertiary Delta. Neutrals Isolated.*—With this connection (Fig. 128), the third-harmonic current will always be confined to the tertiary delta, independent of whether

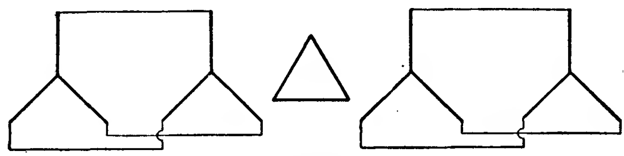

FIG. 128.

the bank is loaded or not. No harmonic, other than the third and its multiples, can flow in the tertiary winding.

Designating the tertiary winding as winding 3 and assuming unity ratio of transformation also for this winding, the following equations may be written

$$E_{1c}''' = E_{2c}''' = E_{3c}''' = I_3'''Z_{33}''' \tag{46}$$

$$V_1''' = E_{1c}''' - jI_3'''X_{13}''' = I_3'''Z_{3(1)}''' \tag{47}$$

$$V_2''' = E_{2c}''' - jI_3'''X_{23}''' = I_3'''Z_{3(2)}''' \tag{48}$$

In equations (47) and (48), $Z_{3(1)}'''$ and $Z_{3(2)}'''$ represent the third-harmonic leakage impedance of the tertiary winding with respect to windings 1 and 2, respectively.

The fifth harmonics behave as in the other Y-Y connections (equations (36), (37), (38), and (39)).

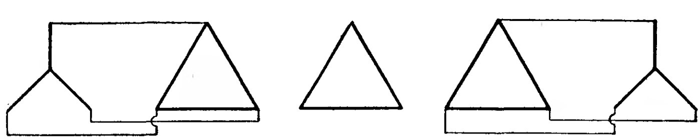

FIG. 129.

(f) Δ-Δ-Δ *Connection.*—With this connection (Fig. 129), the third-harmonic current will circulate in the three deltas under all conditions.

The following equations may be written:

$$E_{1c}''' = (R_1 + jX_{11}''')I_1''' + jX_{12}'''I_2''' + jX_{13}'''I_3''' \quad (49)$$
$$E_{2c}''' = (R_2 + jX_{22}''')I_2''' + jX_{23}'''I_3''' + jX_{21}'''I_1''' \quad (50)$$
$$E_{3c}''' = (R_3 + jX_{33}''')I_3''' + jX_{31}'''I_1''' + jX_{32}'''I_2''' \quad (51)$$
$$I_1''' + I_2''' + I_3''' = I_0''' \quad (52)$$

where I_0''' is the total current required for third-harmonic excitation. In the equations which follow, the triple primes indicating third-harmonic currents and reactances will be omitted. Subtracting equation (50) from equation (49) and equation (51) from equation (50) gives

$$[R_1 + j(X_{11} - X_{21})]I_1 - [R_2 + j(X_{22} - X_{12})]I_2 + j(X_{13} - X_{23})I_3 = 0 \quad (53)$$
$$[R_2 + j(X_{22} - X_{32})]I_2 - [R_3 + j(X_{33} - X_{23})]I_3 + j(X_{21} - X_{31})I_1 = 0 \quad (54)$$

Eliminating I_3 from equation (53) and I_1 from equation (54) by means of equation (52), the following equations are obtained:

$$[R_1 + j(X_{11} - X_{21} - X_{13} + X_{23})]I_1 - [R_2 + j(X_{22} - X_{12} + X_{13} - X_{23})]I_2 + j(X_{13} - X_{23})I_0 = 0 \quad (55)$$
$$[R_2 + j(X_{22} - X_{32} - X_{21} + X_{31})]\, I_2 - [R_3 + j(X_{33} - X_{23} + X_{21} - X_{31})]I_3 + j(X_{21} - X_{31})I_0 = 0 \quad (56)$$

The impedances in the brackets are such as would be assigned to the windings of a three-circuit transformer (or to the branches of its equivalent Y network) when exciting current is neglected (see Chap. II). Hence,

$$Z_1 = R_1 + j(X_{11} - X_{21} - X_{13} + X_{23}) \quad (57)$$
$$Z_2 = R_2 + j(X_{22} - X_{12} + X_{13} - X_{23}) \quad (58)$$
$$Z_3 = R_3 + j(X_{33} - X_{23} + X_{21} - X_{31}) \quad (59)$$

Furthermore,

$$Z_2 - Z_{2(1)} = j(X_{13} - X_{23}) \quad (60)$$
$$Z_2 - Z_{2(3)} = j(X_{13} - X_{12}) \quad (61)$$

where $Z_{2(1)}$ and $Z_{2(3)}$ represent the leakage impedances of winding 2 with respect to winding 1 and winding 3, respectively.

Introducing this abbreviated notation and eliminating I_3 from equation (56) by using equation (52) gives

$$Z_3I_1 + Z_{23}I_2 + (Z_{2(3)} - Z_{23})I_0 = 0 \quad (62)$$

Finally, eliminating I_2 between equations (55) and (62) gives

$$I_1 = \frac{Z_{23}Z_{2(1)} - Z_2Z_{2(3)}}{Z_1Z_{23} + Z_2Z_3}I_0 = \frac{(Z_2 + Z_3)Z_{2(1)} - Z_2Z_{2(3)}}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}I_0 \quad (63)$$

Similarly the currents in winding 2 and 3 become

$$I_2 = \frac{Z_{31}Z_{3(2)} - Z_3Z_{3(1)}}{Z_2Z_{31} + Z_3Z_1}I_0 = \frac{(Z_3 + Z_1)Z_{3(2)} - Z_3Z_{3(1)}}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}I_0 \quad (64)$$

$$I_3 = \frac{Z_{12}Z_{1(3)} - Z_1Z_{1(2)}}{Z_3Z_{12} + Z_1Z_2}I_0 = \frac{(Z_1 + Z_2)Z_{1(3)} - Z_1Z_{1(2)}}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}I_0 \quad (65)$$

From equations (63), (64), and (65), it follows that

$$\frac{I_1}{Z_{23}Z_{2(1)} - Z_2Z_{2(3)}} = \frac{I_2}{Z_{31}Z_{3(2)} - Z_3Z_{3(1)}} = \frac{I_3}{Z_{12}Z_{1(3)} - Z_1Z_{1(2)}} \quad (66)$$

which gives the distribution of third-harmonic current between the three deltas.

As far as the fifth harmonic is concerned, the transformer operates as if only the primary and secondary circuits were present since no fifth-harmonic current can flow in delta 3 under any conditions. Of course, this statement assumes that no external load circuit is connected to the tertiary delta. The fifth-harmonic distribution, therefore, is given by equations (15), (16), (17), and (18)

EXAMPLE 2

Statement of Problem

A generator station (Fig. 130) has two 12,000 kv.-a., 6,600-volt, 60-cycle turbo alternators, Y-connected, operating in parallel and supplying energy

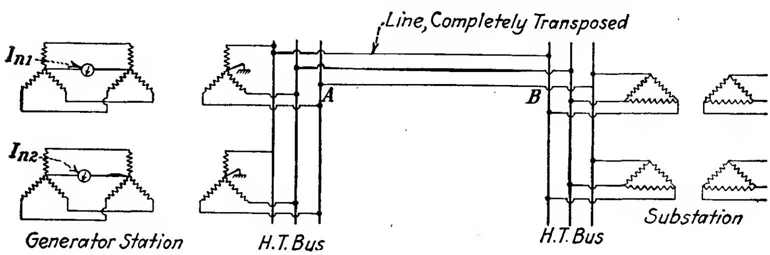

FIG. 130.—Circuit diagram showing generating station and substation connected by completely transposed transmission line (Example 2).

to a 150-kv. transmission line through two banks of Y-Y-connected transformers. The primary neutral points are connected to the neutral points of the generators. The secondary (high-tension) neutrals are grounded.

The wave shape of the generated voltage, both voltage to neutral and voltage between lines, is sinusoidal. The substation transformers are Δ-Δ connected.

The triple-frequency impedance per phase of the generator is $0.04 + j1.30$ ohms. The separate triple-frequency leakage impedances of the transformer windings are

$$\left.\begin{aligned}\text{Low-tension} &= 6.65 + j68.7 \text{ ohms}\\ \text{High-tension} &= 8.43 + j260.4 \text{ ohms}\end{aligned}\right\} \text{per phase referred to high-tension side}$$

With the lines disconnected from the high-tension buses at A and B, the impedance at A of the three *conductors in parallel* to ground has been measured by an impedance bridge and found to be $4 - j540$ ohms at 180 cycles.

With the system in operation, the ammeters in the primary neutrals show

$$\left.\begin{aligned}I_{n1} &= 30.1 \text{ amp.}\\ I_{n2} &= 30.7 \text{ amp.}\end{aligned}\right\} \text{95 per cent assumed to be of triple frequency}$$

Compute the third-harmonic current entering *each conductor* of the three-phase line at A.

Solution

Ratio of Transformation

$$a = \frac{150}{6.6} = 22.72$$

Combining the two parallel circuits, the low-tension triple-frequency impedance per phase referred to the high-tension side becomes

$$\begin{aligned}Z'''_{\text{l.t.}} &= \frac{Z'''_G + Z'''_1}{2} = \frac{1}{2}[(0.04 + j1.30)22.72^2 + 6.65 + j68.7]\\ &= 13.6 + j370.4 = 371 \text{ ohms}\end{aligned}$$

The triple-frequency impedance of the high-tension circuit is

$$\begin{aligned}Z'''_{\text{h.t.}} &= \frac{Z'''_2}{2} + Z'''_{\text{line}} = \frac{1}{2}(8.43 + 260.4) + 3(4 - j540)\\ &= 16.2 - j1{,}490 = 1{,}490 \text{ ohms}\end{aligned}$$

Average primary third-harmonic current per phase of the paralleled circuits referred to the high-tension side

$$I'''_1 = \frac{(30.1 + 30.7) \times 0.95}{3 \times 22.72} = 0.848 \text{ amp.}$$

Third-harmonic current entering *each conductor* of the transmission line

$$I'''_2 = I'''_1 \frac{Z'''_{\text{l.t.}}}{Z'''_{\text{h.t.}}} = \frac{0.848 \times 371}{1{,}490} = 0.211 \text{ amp.}$$

EXAMPLE 3

Statement of Problem

A power station contains four generating units, each of which consists of a three-phase, 12,000 kv.-a., 6,600-volt, 60-cycle turbo alternator (Y-connected) and a bank of three 4,200-kv.-a. transformers. The transformers have three windings (3,800/22,000/86,500 volts) connected Y-Δ-Y and

supply power to a 150-kv. and a 22-kv. transmission line. The line voltages of the generators are strictly sinusoidal.

The 150-kv. line, which is completely transposed, terminates at a substation with Δ-Δ-connected transformers. With the line disconnected from the high-tension buses at power station and substation, the sending-end impedance of the *three conductors in parallel* to ground was measured by an impedance bridge and found to be $20 - j240$ ohms at 180 cycles.

The neutral points of the transformers are grounded on the high-tension side, while the low-tension neutral points, as well as the neutral points of the generators, are isolated.

With all transformers and both lines in operation, the ammeter in each high-tension neutral registers 0.40 amp.

1. Calculate the third-harmonic current in each 22,000-volt winding for this condition.
2. Calculate the third-harmonic current entering the 22,000-volt line.

TRANSFORMER DATA

Winding	Voltage	Ohms resistance referred to high-tension winding
1	3,800	6.65
2	22,000	5.57
3	86,500	8.43

THREE-PHASE TESTS

Transformer connections	Voltage across open Δ		Current in closed Δ	
	Volts	Third harmonic, per cent	Amperes	Third harmonic, per cent
Y-Δ-open Δ (1 2 3)	71.6	62	0.69	99.5
Y-open Δ-Δ (1 2 3)	33.6	91.3	0.18	93

The three-phase data given represent the average of several tests. In every case, the bank was energized from the low-tension side (winding 1) at rated voltage (sinusoidal) and rated frequency.

Solution

Ratio of transformation between the 22,000- and the 86,500-volt windings

$$a_{23} = \frac{86.5}{22} = 3.93$$

Referred to the high-tension side, the triple-frequency separate leakage impedances of winding 2 with respect to 3 and of winding 3 with respect to 2 are as follows:

$$Z_2''' = \frac{71.6 \times 0.62 \times 3.93}{3 \times 0.69 \times 0.995} = 84.6 \text{ ohms}$$

$$Z_3''' = \frac{33.6 \times 0.913 \times 3.93}{3 \times 0.18 \times 0.93} = 240.2 \text{ ohms}$$

$$X_2''' = \sqrt{84.6^2 - 5.57^2} = 84.4 \text{ ohms}$$

$$X_3''' = \sqrt{240.2^2 - 8.43^2} = 240.0 \text{ ohms}$$

$$Z_2''' = 5.57 + j84.4 \text{ ohms}$$

$$Z_3''' = 8.43 + j240.0 \text{ ohms}$$

Equivalent line-to-ground impedance at 180 cycles per transformer per phase

$$= (20 - j240)3 \times 4 = 240 - j2{,}880 \text{ ohms}$$

The equivalent circuit to be used per transformer per phase in determining the division of the third-harmonic current is as shown in Fig. 131.

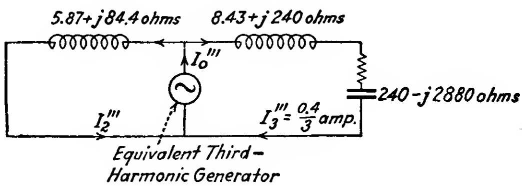

FIG. 131.—Equivalent circuit per transformer per phase, to be used in calculating the division of the third-harmonic current (Example 3).

Total triple-frequency impedance in the high-tension circuit per transformer per phase

$$= 240 - j2{,}880 + 8.43 + j240 = 248.4 - j2{,}640$$
$$= 2{,}652 \text{ ohms}$$

The division of the third-harmonic current is given by

$$\frac{I_2'''}{I_3'''} = \frac{Z_{\text{h.t.}}'''}{Z_2'''}$$

Hence

$$I_2''' = \frac{0.4 \times 2{,}652}{3 \times 84.6} = 4.175 \text{ amp.}$$

1. Actual third-harmonic current circulating in the 22,000-volt windings

$$I_2''' = 4.175 \times 3.93 = 16.4 \text{ amp.}$$

2. No third-harmonic current can enter the 22,000-volt line.

EXAMPLE 4

Statement of Problem

Three 2,100-kv.-a. three-circuit transformers are connected Δ-Δ-Δ. The nominal voltages of the windings are as follows:

Winding 1	2,300 volts.
Winding 2	22,000 volts.
Winding 3	110,000 volts.

The equivalent short-circuit reactances are

6.10 per cent between the 2.3-kv. and the 22-kv. windings.
6.59 per cent between the 2.3-kv. and the 110-kv. windings.
6.15 per cent between the 22-kv. and the 110-kv. windings.

Three of the separate leakage reactances are known, namely,

The reactance of winding 1 with respect to winding 2 = 3.10 per cent
The reactance of winding 2 with respect to winding 3 = 3.00 per cent
The reactance of winding 1 with respect to winding 3 = 3.30 per cent

When 2,300 volts are impressed on winding 1 and the two other windings carry balanced loads, it is found by oscillographic analysis that the primary delta current contains a third-harmonic component of 7.48 amp.

Calculate the third-harmonic currents flowing in the secondary and tertiary windings. Also determine the total third-harmonic magnetizing current, giving the latter in per cent of full-load current.

Solution

The distribution of third-harmonic currents is given by equation (66). Since resistances will be neglected, reactances at fundamental frequency may be used, since changing to triple-frequency reactances merely involves a factor of 3 which would cancel out in equation (66).

The given reactances are

$$\begin{aligned} Z_{12} &= 6.10 \text{ per cent} \\ Z_{23} &= 6.15 \text{ per cent} \\ Z_{31} &= 6.59 \text{ per cent} \\ Z_{1(2)} &= 3.10 \text{ per cent} \\ Z_{2(3)} &= 3.00 \text{ per cent} \\ Z_{1(3)} &= 3.39 \text{ per cent} \end{aligned}$$

The other reactances required in the solution may be calculated from the ones given above.

$$Z_1 = \frac{6.10 + 6.59 - 6.15}{2} = 3.27 \text{ per cent}$$

$$Z_2 = \frac{6.15 + 6.10 - 6.59}{2} = 2.83 \text{ per cent}$$

$$Z_3 = \frac{6.59 + 6.15 - 6.10}{2} = 3.32 \text{ per cent}$$

$$\begin{aligned} Z_{2(1)} &= 6.10 - 3.10 = 3.00 \text{ per cent} \\ Z_{3(2)} &= 6.15 - 3.00 = 3.15 \text{ per cent} \\ Z_{3(1)} &= 6.59 - 3.39 = 3.20 \text{ per cent} \end{aligned}$$

Evaluating the denominators in equation (66) gives

$$\begin{aligned} Z_{23}Z_{2(1)} - Z_2Z_{2(3)} &= 6.15 \times 3.00 - 2.83 \times 3.00 = 9.96 \text{ per cent} \\ Z_{31}Z_{3(2)} - Z_3Z_{3(1)} &= 6.59 \times 3.15 - 3.32 \times 3.20 = 10.13 \text{ per cent} \\ Z_{12}Z_{1(3)} - Z_1Z_{1(2)} &= 6.10 \times 3.39 - 3.27 \times 3.10 = 10.55 \text{ per cent} \end{aligned}$$

The 100 per cent current referred to the primary is

$$I = \frac{2,100}{2.3} = 913 \text{ amp.}$$

In percentage of the full-load current, the third-harmonic currents become

$$I_1''' = \frac{7.48 \times 100}{913} = 0.82 \text{ per cent}$$

$$I_2''' = \frac{0.82 \times 10.13}{9.96} = 0.833 \text{ per cent}$$

$$I_3''' = \frac{0.82 \times 10.55}{9.96} = 0.867 \text{ per cent}$$

The total third-harmonic magnetizing current is, hence,

$$I_0''' = 0.82 + 0.833 + 0.867 = 2.52 \text{ per cent}$$

The actual third-harmonic currents flowing in the 22- and 110-kv. windings are

$$I_2''' = \frac{913 \times 0.833 \times 2.3}{100 \times 22} = 0.795 \text{ amp.}$$

$$I_3''' = \frac{913 \times 0.867 \times 2.3}{100 \times 110} \doteq 0.166 \text{ amp.}$$

CHAPTER IX

GENERAL CIRCUIT CONSTANTS

Conception of General Circuit Constants.—In a network consisting of any combination of constant impedances and in which power is supplied at one point and received at another point, the voltage and current at each point can, in the steady state, be expressed in terms of the voltage and current at the other by very simple relations. Consider the network shown in Fig. 132,

FIG. 132.

for instance, and let E_s, E_r, I_s, and I_r denote the voltages and currents at the sending and receiving ends, respectively. The equations interrelating these quantities may then be written

$$E_s = A_0E_r + B_0I_r \tag{1}$$
$$I_s = C_0E_r + D_0I_r \tag{2}$$

Solving these equations[1] with respect to the receiver voltage and current, the latter are given by

$$E_r = D_0E_s - B_0I_s \tag{3}$$
$$I_r = -C_0E_s + A_0I_s \tag{4}$$

A_0, B_0, C_0, and D_0 are all constants, complex quantities in the general case, and depend upon the network impedances. They are called *general circuit constants.*

It has long been known that the simple relations stated above existed in a network for the conditions specified. It is believed, however, that Evans and Sels[2] were the first ones to call attention to the great advantage of using general circuit constants in the

[1] Note that, in establishing equations (3) and (4) from equations (1) and (2), use is made of the relation given by equation (26).

[2] EVANS, R. D., and SELS, H. K., "Transmission-line Constants and Resonance," *Elec. Jour.*, p. 306, July, 1921, and "Transmission Lines and Transformers," *Elec. Jour.*, p. 356, August, 1921.

solution of many problems. They are particularly convenient when it comes to combination of networks and also in the calculation of circle diagrams for graphical solution of transmission problems.

Determination of General Circuit Constants.—The general circuit constants can always be determined by working out expressions for the sending voltage and current in terms of the receiving voltage and current (or *vice versa*), making use of the original impedances and admittances which make up the network. The terms in these expressions may then be collected so that equations of the form (1) to (4) result. The general circuit constants are then written down by inspection.

In many cases, the establishment of the expressions for the general circuit constants is both complicated and laborious. If formulas, once for all, are worked out, however, and tabulated for the circuits most frequently met with in practice, which quite readily can be done as far as transmission circuits are concerned, then the appropriate formula may in many cases be selected directly from the table. Furthermore, by making proper use of the possibility of series and multiple combination of networks by pre-established formulas, numerous network configurations may be worked out with comparative ease, even if the formulas for the particular configuration are not included in the table.

1. *Lumped Impedance.*—For the lumped impedance (Fig. 133), the following equations evidently hold:

$$E_s = E_r + ZI_r \tag{5}$$
$$I_s = I_r \tag{6}$$

Hence, for this circuit,

$$A = 1, \quad B = Z, \quad C = 0, \quad D = 1 \tag{7}$$

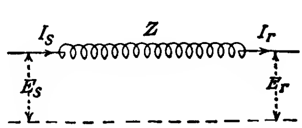

FIG. 133.—Lumped impedance.

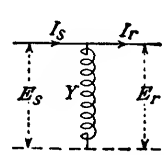

FIG. 134.—Lumped admittance.

2. *Lumped Admittance.*—Considering the lumped admittance (Fig. 134), the following equations may be written:

$$E_s = E_r \tag{8}$$
$$I_s = YE_r + I_r \tag{9}$$

Hence,

$$A = 1, \quad B = 0, \quad C = Y, \quad D = 1 \tag{10}$$

3. Π-*circuit.*—In order to cover the general case, a dissymmetrical Π-circuit, as shown in Fig. 135, will first be considered. For this circuit, the following relations hold:

$$E_s = (1 + ZY_2)E_r + ZI_r \tag{11}$$
$$I_s = (Y_1 + Y_2 + ZY_1Y_2)E_r + (1 + ZY_1)I_r \tag{12}$$

Hence, for the dissymmetrical π,

$$\begin{aligned} A &= 1 + ZY_2 & B &= Z \\ C &= Y_1 + Y_2 + ZY_1Y_2 & D &= 1 + ZY_1 \end{aligned} \tag{13}$$

If the Π-circuit is symmetrical, the admittances of the two leaks are equal and may be put equal to $Y/2$. Thus, the general circuit constants for a symmetrical Π become

$$\begin{aligned} A &= 1 + \frac{ZY}{2} & B &= Z \\ C &= Y\left(1 + \frac{ZY}{4}\right) & D &= 1 + \frac{ZY}{2} \end{aligned} \tag{14}$$

It is interesting to note that the constant B is always equal to the architrave impedance of the Π-circuit whether the latter is symmetrical or not.

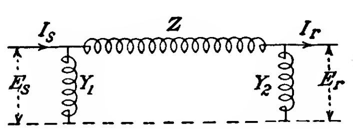

Fig. 135.—Dissymmetrical Π-circuit.

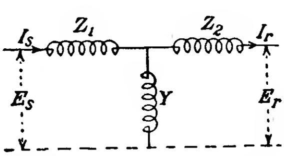

Fig. 136.—Dissymmetrical T-circuit.

4. *T-circuit.*—Considering first the dissymmetrical T-circuit (Fig. 136), the general relations may be written

$$E_s = (1 + Z_1Y)E_r + (Z_1 + Z_2 + Z_1Z_2Y)I_r \tag{15}$$
$$I_s = YE_r + (1 + Z_2Y)I_r \tag{16}$$

Hence, for the dissymmetrical T,

$$\begin{aligned} A &= 1 + Z_1Y & B &= Z_1 + Z_2 + Z_1Z_2Y \\ C &= Y & D &= 1 + Z_2Y \end{aligned} \tag{17}$$

If the T-circuit is symmetrical, the impedances of the two arms are equal and may be put equal to $Z/2$. The general circuit constants of the symmetrical T are given by

$$\begin{aligned} A &= 1 + \frac{ZY}{2} & B &= Z\left(1 + \frac{ZY}{4}\right) \\ C &= Y & D &= 1 + \frac{ZY}{2} \end{aligned} \tag{18}$$

In connection with the T-circuit, it is of interest to note that the constant C is always equal to the admittance of the T, independent of whether or not the latter is symmetrical.

5. *The Transformer.*—As previously demonstrated, the equivalent circuit of a two-winding transformer is in general a dissymmetrical T, as indicated in Fig. 136. The impedances of the branches are equal to the separate leakage impedances of the two windings respectively, *viz.*,

$$Z_1 = R_1 + jX_1 \tag{19}$$
$$Z_2 = R_2 + jX_2 \tag{20}$$

and should be determined by tests which correctly give the separate leakage impedances. Often no such test data are available. Most frequently, therefore, they are assumed equal when referred to the same side, each being one-half the equivalent impedance of the transformer. As also previously discussed in considerable detail, the admittance of the leak is usually taken as the apparent open-circuit admittance of the transformer at rated impressed voltage. Hence the admittance used is given by

$$Y = \frac{1}{R_1 + R_c + j(X_1 + X_c)} = \frac{1}{R_1 + R_c + j\omega L_1} \tag{21}$$

Since R_1 and X_1 are small compared to R_c and X_c, this admittance is very nearly equal to the correct admittance

$$Y = \frac{1}{R_c + jX_c} = \frac{1}{R_c + j\omega M_{12}} \tag{22}$$

It is unnecessary to determine this admittance with very great accuracy, since it varies with the saturation and, hence, depends upon the voltage.[1]

The general circuit constants of the transformer are, therefore, given by equations (17) or (18), as the case may be. If the transformer is approximately represented by a Π-circuit, which evi-

[1] See Chap. II, pp. 39 to 42 inclusive.

dently is possible, equation (13) or (14) should be used for the general circuit constants. Sometimes the transformer is represented by a *cantilever* circuit; *i.e.*, the total admittance is connected at one of the terminals and the general circuit constants should then be determined for this circuit.

As a matter of fact, although the T representation of the transformer is the logical and fundamentally correct one, it makes very little difference in practice which representation is used unless unusual refinements are required.

6. *Transmission Line.*—The relation between the voltages and currents at the terminals of a long transmission line is given by

$$E_s = E_r \cosh \theta + I_r Z_0 \sinh \theta \tag{23}$$

$$I_s = E_r \frac{\sinh \theta}{Z_0} + I_r \cosh \theta \tag{24}$$

Hence, the general circuit constants of the transmission line are

$$\begin{aligned} A &= \cosh \theta \quad & B &= Z_0 \sinh \theta \\ C &= \frac{\sinh \theta}{Z_0} \quad & D &= \cosh \theta \end{aligned} \tag{25}$$

Two Important Relations.—When a circuit is symmetrical, the general circuit constants A and D are always equal. Inspection of the formulas for the Π-circuit, for instance, shows it to be true for this particular circuit. Since, however, any symmetrical circuit is reducible to a symmetrical equivalent Π, the relation just stated must hold in the general symmetrical case.

It is seen to hold for the transmission line, where $A = D = \cosh \theta$. Furthermore, a transmission line with identical sending and receiving transformers will also have equal A_0 and D_0 constants. On the other hand, if the sending and receiving transformers are different, the general circuit constants A_0 and D_0 of the whole system will be unequal in spite of the fact that each of the three parts comprising the system may in itself be a symmetrical circuit.

A second relation which is generally true is the following:

$$AD - BC = 1 \tag{26}$$

Resorting to the dissymmetrical Π, equation (26) may be proved for the general case. Forming the products involved by using equation (13) gives

$$AD = (1 + ZY_2)(1 + ZY_1) = 1 + ZY_1 + ZY_2 + Z^2Y_1Y_2 \tag{27}$$

$$BC = Z(Y_1 + Y_2 + ZY_1Y_2) = ZY_1 + ZY_2 + Z^2Y_1Y_2 \tag{28}$$

Subtraction immediately gives equation (26).

Networks in Series and Multiple.—Any number of networks connected in series or multiple may be represented by a single set of general circuit constants between any two terminal points, provided power is supplied or received at these points only. If power is supplied or received at other points, representation as stated is not possible, unless the effect of power supplied or received at the intermediate points can be taken into account by the *insertion of a constant impedance at these points.* There are but few cases, however, where this can be done, since the equivalent impedances representing generating stations or loads in general would depend on the voltage.

Below, a few cases of series and multiple networks will be treated. It should be carefully noted that in determining the general circuit constants of a series network *the order of the series sections must, in general, not be changed.*

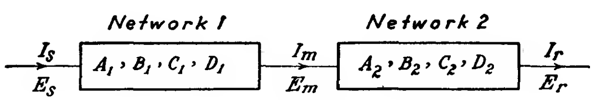

FIG. 137.—Two networks in series.

1. *Two Networks in Series.*—Let the separate networks shown in Fig. 137 have the general circuit constants A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2. The following equations hold:

$$E_s = A_1E_m + B_1I_m \tag{29}$$
$$I_s = C_1E_m + D_1I_m \tag{30}$$
$$E_m = A_2E_r + B_2I_r \tag{31}$$
$$I_m = C_2E_r + D_2I_r \tag{32}$$

Substituting equations (31) and (32) in equations (29) and (30) and collecting terms gives

$$E_s = (A_1A_2 + B_1C_2)E_r + (A_1B_2 + B_1D_2)I_r \tag{33}$$
$$I_s = (C_1A_2 + D_1C_2)E_r + (C_1B_2 + D_1D_2)I_r \tag{34}$$

Hence, the general circuit constants of two networks in series are

$$\left.\begin{aligned} A_0 &= A_1A_2 + B_1C_2 \\ B_0 &= A_1B_2 + B_1D_2 \\ C_0 &= C_1A_2 + D_1C_2 \\ D_0 &= C_1B_2 + D_1D_2 \end{aligned}\right\} \tag{35}$$

2. *Three Networks in Series.*—Figure 138 shows this case. The following equations may immediately be written down:

$$E_s = A_1E_m + B_1I_m \tag{36}$$
$$I_s = C_1E_m + D_1I_m \tag{37}$$
$$E_m = A_2E_n + B_2I_n \tag{38}$$
$$I_m = C_2E_n + D_2I_n \tag{39}$$
$$E_n = A_3E_r + B_3I_r \tag{40}$$
$$I_n = C_3E_r + D_3I_r \tag{41}$$

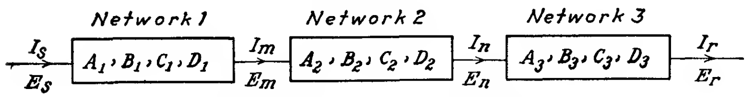

FIG. 138.—Three networks in series.

Substituting equations (40) and (41) in equations (38) and (39) gives

$$E_m = (A_2A_3 + B_2C_3)E_r + (A_2B_3 + B_2D_3)I_r \tag{42}$$
$$I_m = (C_2A_3 + D_2C_3)E_r + (C_2B_3 + D_2D_3)I_r \tag{43}$$

Substituting equations (42) and (43) in equations (36) and (37) and collecting terms gives

$$E_s = [A_1(A_2A_3 + B_2C_3) + B_1(C_2A_3 + D_2C_3)]E_r + [A_1(A_2B_3 + B_2D_3) + B_1(C_2B_3 + D_2D_3)]I_r \tag{44}$$
$$I_s = [C_1(A_2A_3 + B_2C_3) + D_1(C_2A_3 + D_2C_3)]E_r + [C_1(A_2B_3 + B_2D_3) + D_1(C_2B_3 + D_2D_3)]I_r \tag{45}$$

Hence, the general circuit constants of three networks in series are

$$\left.\begin{aligned} A_0 &= A_1(A_2A_3 + B_2C_3) + B_1(C_2A_3 + D_2C_3) \\ B_0 &= A_1(A_2B_3 + B_2D_3) + B_1(C_2B_3 + D_2D_3) \\ C_0 &= C_1(A_2A_3 + B_2C_3) + D_1(C_2A_3 + D_2C_3) \\ D_0 &= C_1(A_2B_3 + B_2D_3) + D_1(C_2B_3 + D_2D_3) \end{aligned}\right\} \tag{46}$$

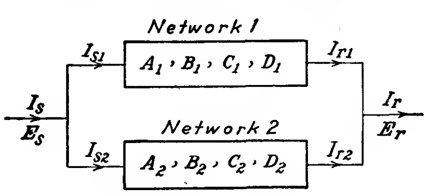

FIG. 139.—Two networks in parallel.

3. *Two Networks in Parallel.*—Considering the two networks in parallel, as shown in Fig. 139, the following equations may be written down immediately.

$$E_s = A_1E_r + B_1I_{r1} \tag{47}$$
$$E_s = A_2E_r + B_2I_{r2} \tag{48}$$
$$I_{s1} = C_1E_r + D_1I_{r1} \tag{49}$$
$$I_{s2} = C_2E_r + D_2I_{r2} \tag{50}$$
$$I_s = I_{s1} + I_{s2} \tag{51}$$
$$I_r = I_{r1} + I_{r2} \tag{52}$$

Substituting equation (52) in equation (47) gives

$$E_s = A_1E_r + B_1I_r - B_1I_{r2} \tag{53}$$

Eliminating I_{r2} between equations (48) and (53) gives, upon contraction,

$$E_s = \frac{A_1B_2 + B_1A_2}{B_1 + B_2}E_r + \frac{B_1B_2}{B_1 + B_2}I_r \tag{54}$$

From equations (49), (50), and (51) is obtained

$$I_s = (C_1 + C_2)E_r + D_1I_{r1} + D_2I_{r2} \tag{55}$$

which, combined with equations (52) and (48) gives

$$I_s = (C_1 + C_2)E_r + D_1I_r + \frac{D_2 - D_1}{B_2}E_s - \frac{A_2(D_2 - D_1)}{B_2}E_r \tag{56}$$

Substituting for E_s from equation (54) and contracting gives

$$I_s = \left[C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2}\right]E_r + \frac{B_1D_2 + D_1B_2}{B_1 + B_2}I_r \tag{57}$$

From equations (54) and (57), the general circuit constants for two networks in parallel are obtained as

$$\left.\begin{aligned}
A_0 &= \frac{A_1B_2 + B_1A_2}{B_1 + B_2} \\
B_0 &= \frac{B_1B_2}{B_1 + B_2} \\
C_0 &= C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \\
D_0 &= \frac{B_1D_2 + D_1B_2}{B_1 + B_2}
\end{aligned}\right\} \tag{58}$$

It is useful to note that, when the two networks operated in parallel are identical, the general circuit constants become

$$\begin{aligned}
A_0 &= A \quad & B_0 &= \frac{B}{2} \\
C_0 &= 2C \quad & D_0 &= D
\end{aligned} \tag{58a}$$

Hence, the constants A_0 and D_0 are equal to, the constant B_0 one-half, and C_0 twice the corresponding constants of each network separately.

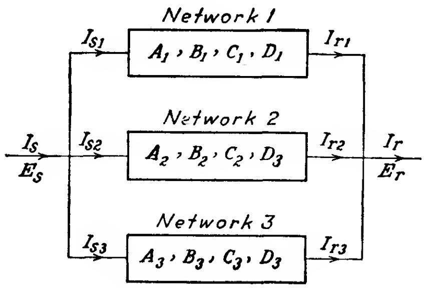

FIG. 140.—Three networks in parallel.

4. *Three Networks in Parallel.*—Figure 140 shows three networks in parallel. The general circuit constants of this combination can be obtained from the following relations:

$$E_s = A_1E_r + B_1I_{r1} \tag{59}$$
$$E_s = A_2E_r + B_2I_{r2} \tag{60}$$
$$E_s = A_3E_r + B_3I_{r3} \tag{61}$$
$$I_{s1} = C_1E_r + D_1I_{r1} \tag{62}$$
$$I_{s2} = C_2E_r + D_2I_{r2} \tag{63}$$
$$I_{s3} = C_3E_r + D_3I_{r3} \tag{64}$$
$$I_s = I_{s1} + I_{s2} + I_{s3} \tag{65}$$
$$I_r = I_{r1} + I_{r2} + I_{r3} \tag{66}$$

Substituting equation (66) in equation (59) gives

$$E_s = A_1E_r + B_1I_r - B_1I_{r2} - B_1I_{r3} \tag{67}$$

Eliminating I_{r2} and I_{r3} by making use of equations (60) and (61) gives

$$E_s = A_1E_r + B_1I_r - \frac{B_1}{B_2}E_s + \frac{B_1A_2}{B_2}E_r - \frac{B_1}{B_3}E_s + \frac{B_1A_3}{B_3}E_r \tag{68}$$

The solution of this equation with respect to E_s is

$$E_s = \frac{A_1B_2B_3 + B_1A_2B_3 + B_1B_2A_3}{B_1B_2 + B_2B_3 + B_1B_3}E_r + \frac{B_1B_2B_3}{B_1B_2 + B_2B_3 + B_1B_3}I_r \tag{69}$$

From equations (62), (63), (64), and (65) is obtained

$$I_s = (C_1 + C_2 + C_3)E_r + D_1I_{r1} + D_2I_{r2} + D_3I_{r3} \tag{70}$$

which, when combined with equations (66), (60), and (61) becomes

$$I_s = (C_1 + C_2 + C_3)E_r + D_1I_r + \frac{D_2 - D_1}{B_2}E_s - \frac{(D_2 - D_1)A_2}{B_2}E_r + \frac{D_3 - D_1}{B_3}E_s - \frac{(D_3 - D_1)A_3}{B_3}E_r \quad (71)$$

Substituting for E_s from equation (69) and reducing, the following expression results:

$$I_s = \left[C_1 + C_2 + C_3 + \frac{(A_1-A_2)(D_2-D_1)B_3+(A_2-A_3)(D_3-D_2)B_1+(A_3-A_1)(D_1-D_3)B_2}{B_1B_2 + B_2B_3 + B_1B_3}\right]E_r + \frac{D_1B_2B_3 + D_2B_1B_3 + D_3B_1B_2}{B_1B_2 + B_2B_3 + B_1B_3}I_r \quad (72)$$

Inspection of equations (69) and (72) gives the general circuit constants of three parallel networks, as

$$\left.\begin{aligned}
A_0 &= \frac{A_1B_2B_3 + B_1A_2B_3 + B_1B_2A_3}{B_1B_2 + B_2B_3 + B_1B_3} \\
B_0 &= \frac{B_1B_2B_3}{B_1B_2 + B_2B_3 + B_1B_3} \\
C_0 &= C_1 + C_2 + C_3 + \frac{(A_1-A_2)(D_2-D_1)B_3+(A_2-A_3)(D_3-D_2)B_1+(A_3-A_1)(D_1-D_3)B_2}{B_1B_2 + B_2B_3 + B_1B_3} \\
D_0 &= \frac{D_1B_2B_3 + D_2B_1B_3 + D_3B_1B_2}{B_1B_2 + B_2B_3 + B_1B_3}
\end{aligned}\right\} \quad (73)$$

If the three networks are identical, the general circuit constants are given by

$$\begin{aligned} A_0 &= A \quad & B_0 &= \frac{B}{3} \\ C_0 &= 3C \quad & D_0 &= D \end{aligned} \quad (73a)$$

Hence, an equally simple relation holds as in the case of two parallel identical networks. It may now be inferred that, if n identical networks are operated in parallel, the general circuit constants would be

$$\begin{aligned} A_0 &= A \quad & B_0 &= \frac{B}{n} \\ C_0 &= nC \quad & D_0 &= D \end{aligned} \quad (73b)$$

Combination of Networks Usually Met With in Power Transmission.[1]—The transmission circuit proper consists of lines and transformers. The combinations which most frequently present themselves for solution are line with sending transformers, line with receiving transformers, and line with both sending and receiving transformers. The general circuit constants for these will be given below.

The transformers will be replaced by their symmetrical equivalent T-circuits, the leakage impedance of the two windings being considered equal. Exact values of the general circuit constants of the combinations are obtained when this transformer circuit is used. The formal expressions for the general circuit constants, however, are greatly simplified when the transformers are represented by their approximate "cantilever" circuits. Formulas worked out on this basis are also given. The error introduced by the use of the simplified formulas is rather insignificant[2] and considerable time and labor are saved in the computations. Subscripts s attached to the transformer impedances and admittances signify sending transformers, subscripts r receiving transformers.

A, B, C, and D will represent the general circuit constants of the line itself, whether the latter consists of a single line, two or more equal lines in parallel, two or more unequal lines in parallel or a sectionalized line with one or more sections out of operation.

Transformers Represented by Equivalent T-circuits. 1. *Transmission Line with Sending Transformers (Exact).*—Figure

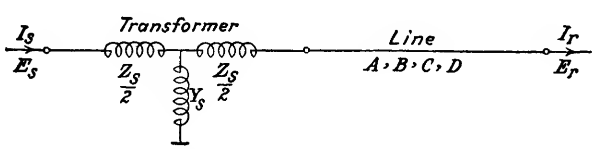

FIG. 141.—Transmission line with sending transformers. Transformers represented by equivalent T-circuit.

141 shows this circuit. By using the formulas previously given for the transformer and for the combination of two networks in series, the general circuit constants for the combination become

[1] EVANS and SELS in their paper "Transmission Lines and Transformers," *loc. cit.*, give a very complete table of formal expressions for the general circuit constants of various networks and network combinations.

[2] EVANS and SELS in their paper, *loc. cit.*, state that the approximate error with this circuit is about 0.5 per cent.

$$\left.\begin{aligned}
A_0 &= A\left(1 + \frac{Z_sY_s}{2}\right) + CZ_s\left(1 + \frac{Z_sY_s}{4}\right)\\
B_0 &= B\left(1 + \frac{Z_sY_s}{2}\right) + DZ_s\left(1 + \frac{Z_sY_s}{4}\right)\\
C_0 &= C\left(1 + \frac{Z_sY_s}{2}\right) + AY_s\\
D_0 &= D\left(1 + \frac{Z_sY_s}{2}\right) + BY_s
\end{aligned}\right\} \quad (74)$$

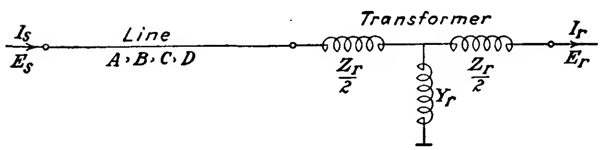

FIG. 142.—Transmission line with receiving transformers. Transformers represented by equivalent T-circuit.

2. *Transmission Line with Receiving Transformers (Exact).*—This circuit is shown in Fig. 142. By a process as in the preceding case, the general circuit constants are obtained as

$$\left.\begin{aligned}
A_0 &= A\left(1 + \frac{Z_rY_r}{2}\right) + BY_r\\
B_0 &= B\left(1 + \frac{Z_rY_r}{2}\right) + AZ_r\left(1 + \frac{Z_rY_r}{4}\right)\\
C_0 &= C\left(1 + \frac{Z_rY_r}{2}\right) + DY_r\\
D_0 &= D\left(1 + \frac{Z_rY_r}{2}\right) + CZ_r\left(1 + \frac{Z_rY_r}{4}\right)
\end{aligned}\right\} \quad (75)$$

3. *Transmission Line with Sending and Receiving Transformers (Exact).*—Figure 143 shows this circuit. By inserting in the

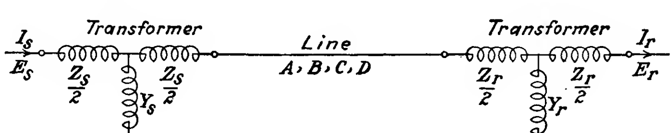

FIG. 143.—Transmission line with sending and receiving transformers. Transformers represented by equivalent T-circuits.

formulas for three networks in series the general circuit constants of this case become

$$
\left.\begin{aligned}
A_0 &= \left(1 + \frac{Z_sY_s}{2}\right)\left[A\left(1 + \frac{Z_rY_r}{2}\right) + BY_r\right] + \\
&\qquad Z_s\left(1 + \frac{Z_sY_s}{4}\right)\left[C\left(1 + \frac{Z_rY_r}{2}\right) + DY_r\right] \\
B_0 &= \left(1 + \frac{Z_sY_s}{2}\right)\left[AZ_r\left(1 + \frac{Z_rY_r}{4}\right) + B\left(1 + \frac{Z_rY_r}{2}\right)\right] + \\
&\qquad Z_s\left(1 + \frac{Z_sY_s}{4}\right)\left[CZ_r\left(1 + \frac{Z_rY_r}{4}\right) + D\left(1 + \frac{Z_rY_r}{2}\right)\right] \\
C_0 &= Y_s\left[A\left(1 + \frac{Z_rY_r}{2}\right) + BY_r\right] + \\
&\qquad \left(1 + \frac{Z_sY_s}{2}\right)\left[C\left(1 + \frac{Z_rY_r}{2}\right) + DY_r\right] \\
D_0 &= Y_s\left[AZ_r\left(1 + \frac{Z_rY_r}{4}\right) + B\left(1 + \frac{Z_rY_r}{2}\right)\right] + \\
&\qquad \left(1 + \frac{Z_sY_s}{2}\right)\left[CZ_r\left(1 + \frac{Z_rY_r}{4}\right) + D\left(1 + \frac{Z_rY_r}{2}\right)\right]
\end{aligned}\right\} \quad (76)
$$

If, in the above combinations, the transmission line is symmetrical, *i.e.*, consists of a single line only or identical lines in parallel, the constants A and D are equal and the formulas given may be somewhat simplified. Thus, for a symmetrical line, equations (76) may be written

$$
\left.\begin{aligned}
A_0 &= A\left[\left(1 + \frac{Z_sY_s}{2}\right)\left(1 + \frac{Z_rY_r}{2}\right) + Y_rZ_s\left(1 + \frac{Z_sY_s}{4}\right)\right] + \\
&\qquad BY_r\left(1 + \frac{Z_sY_s}{2}\right) + CZ_s\left(1 + \frac{Z_sY_s}{4}\right)\left(1 + \frac{Z_rY_r}{2}\right) \\
B_0 &= B\left(1+\frac{Z_sY_s}{2}\right)\left(1+\frac{Z_rY_r}{2}\right)+A\left[Z_r\left(1+\frac{Z_rY_r}{4}\right)\left(1+\frac{Z_sY_s}{2}\right)\right. \\
&\quad \left. + Z_s\left(1+\frac{Z_sY_s}{4}\right)\left(1+\frac{Z_rY_r}{2}\right)\right]+CZ_sZ_r\left(1+\frac{Z_sY_s}{4}\right)\left(1+\frac{Z_rY_r}{4}\right) \\
C_0 &= C\left(1 + \frac{Z_sY_s}{2}\right)\left(1 + \frac{Z_rY_r}{2}\right) + A\left[Y_r\left(1 + \frac{Z_sY_s}{2}\right) + \right. \\
&\qquad \left. Y_s\left(1 + \frac{Z_rY_r}{2}\right)\right] + BY_sY_r \\
D_0 &= A\left[\left(1 + \frac{Z_sY_s}{2}\right)\left(1 + \frac{Z_rY_r}{2}\right) + Y_sZ_r\left(1 + \frac{Z_rY_r}{4}\right)\right] + \\
&\qquad BY_s\left(1 + \frac{Z_rY_r}{2}\right) + CZ_r\left(1 + \frac{Z_rY_r}{4}\right)\left(1 + \frac{Z_sY_s}{2}\right)
\end{aligned}\right\} \quad (77)
$$

If, in addition to the lines being symmetrical, the sending and receiving transformers are identical, equations (77) reduce to

$$\left.\begin{aligned}
A_0 &= A\left[\left(1+\frac{ZY}{2}\right)^2 + ZY\left(1+\frac{ZY}{4}\right)\right] + BY\left(1+\frac{ZY}{2}\right) + \\
&\qquad CZ\left(1+\frac{ZY}{2}\right)\left(1+\frac{ZY}{4}\right) \\
B_0 &= B\left(1+\frac{ZY}{2}\right)^2 + 2AZ\left(1+\frac{ZY}{2}\right)\left(1+\frac{ZY}{4}\right) + \\
&\qquad CZ^2\left(1+\frac{ZY}{4}\right)^2 \\
C_0 &= C\left(1+\frac{ZY}{2}\right)^2 + 2AY\left(1+\frac{ZY}{2}\right) + BY^2 \\
D_0 &= A\left[\left(1+\frac{ZY}{2}\right)^2 + ZY\left(1+\frac{ZY}{4}\right)\right] + BY\left(1+\frac{ZY}{2}\right) + \\
&\qquad CZ\left(1+\frac{ZY}{2}\right)\left(1+\frac{ZY}{4}\right)
\end{aligned}\right\} \quad (78)$$

Transformers Represented by Cantilever Circuits. 1. *Transmission Line with Sending Transformers (Approximate).*—Figure

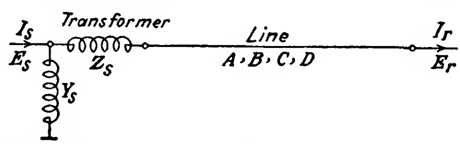

FIG. 144.—Transmission line with sending transformers. Transformers represented by cantilever circuit (approximate).

144 shows the approximate circuit. Using the formulas for two networks in series the general circuit constants become

$$\left.\begin{aligned}
A_0 &= A + CZ_s \\
B_0 &= B + DZ_s \\
C_0 &= C(1 + Z_sY_s) + AY_s \\
D_0 &= D(1 + Z_sY_s) + BY_s
\end{aligned}\right\} \quad (79)$$

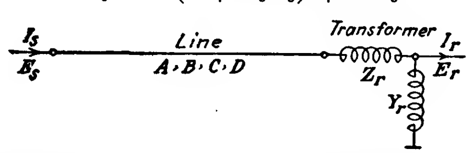

FIG. 145.—Transmission line with receiving transformers. Transformers represented by cantilever circuit (approximate).

2. *Transmission Line with Receiving Transformers (Approximate).*—The approximate circuit is shown in Fig. 145. The general circuit constants are

$$\left.\begin{aligned} A_0 &= A(1 + Z_rY_r) + BY_r \\ B_0 &= B + AZ_r \\ C_0 &= C(1 + Z_rY_r) + DY_r \\ D_0 &= D + CZ_r \end{aligned}\right\} \quad (80)$$

3. *Transmission Line with Sending and Receiving Transformers (Approximate).*—Figure 146 shows the approximate circuit.

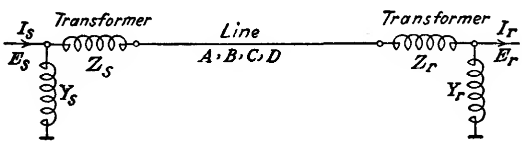

FIG. 146.—Transmission line with sending and receiving transformers. Transformers represented by cantilever circuits (approximate).

Using the formulas for three networks in series, the general circuit constants are obtained as

$$\left.\begin{aligned} A_0 &= A(1 + Z_rY_r) + BY_r + CZ_s(1 + Z_rY_r) + DZ_sY_r \\ B_0 &= AZ_r + B + CZ_sZ_r + DZ_s \\ C_0 &= AY_s(1 + Z_rY_r) + BY_sY_r + C(1 + Z_sY_s)(1 + Z_rY_r) + \\ &\qquad DY_r(1 + Z_sY_s) \\ D_0 &= AZ_rY_s + BY_s + CZ_r(1 + Z_sY_s) + D(1 + Z_sY_s) \end{aligned}\right\} \quad (81)$$

If, in this case, the transmission line is symmetrical so that A and D are equal, the above formulas reduce to

$$\left.\begin{aligned} A_0 &= A[(1 + Z_rY_r) + Z_sY_r] + BY_r + CZ_s(1 + Z_rY_r) \\ B_0 &= A(Z_s + Z_r) + B + CZ_sZ_r \\ C_0 &= A[Y_s(1 + Z_rY_r) + Y_r(1 + Z_sY_s)] + BY_sY_r + \\ &\qquad C(1 + Z_sY_s)(1 + Z_rY_r) \\ D_0 &= A[(1 + Z_sY_s) + Z_rY_s] + BY_s + CZ_r(1 + Z_sY_s) \end{aligned}\right\} \quad (82)$$

If, in addition to the line's being symmetrical, the sending and receiving transformers are identical, the expressions for the general circuit constants may be simplified still more, as follows:

$$\left.\begin{aligned} A_0 &= A(1 + 2ZY) + BY + CZ(1 + ZY) \\ B_0 &= A2Z + B + CZ^2 \\ C_0 &= A2Y(1 + ZY) + BY^2 + C(1 + ZY)^2 \\ D_0 &= A(1 + 2ZY) + BY + CZ(1 + ZY) \end{aligned}\right\} \quad (83)$$

EXAMPLE 1

Statement of Problem

Calculate the general circuit constants of a 90-mile, three-phase, two-circuit, 220-kv., 60-cycle transmission line with a 150,000-kv.-a. bank of receiving transformers (see Fig. 147). The constants of line and transformers are given below.

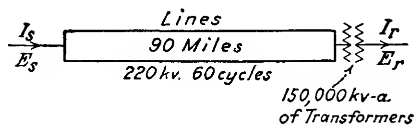

FIG. 147.—Single-line diagram of two parallel three-phase transmission lines with receiving transformers (Example 1).

Solution

Since the two transmission circuits are symmetrical, the constants will be calculated for *one circuit in connection with one-half the transformer capacity.*

Constants of line:

90 miles, 605,000 cir. mils, A.C.S.R.

$$R = 0.154 \text{ ohm/mile}$$
$$X = 0.853 \text{ ohm/mile}$$
$$G = 0.1 \times 10^{-6} \text{ mho/mile}$$
$$C = 0.01352 \times 10^{-6} \text{ farad/mile}$$
$$z = 0.154 + j0.853 = 0.866\underline{/79°.77} \text{ ohm/mile}$$
$$y = (0.1 + j377 \times 0.01352)10^{-6} = (0.1 + j5.1)10^{-6} = 5.1 \times 10^{-6}\underline{/88°.88} \text{ mho/mile}$$
$$\alpha = \sqrt{zy} = 2.102 \times 10^{-3}\underline{/84°.32} \text{ hyp/mile}$$
$$Z_0 = \sqrt{\frac{z}{y}} = 412\overline{\backslash 4°.55} \text{ ohms}$$
$$\theta = \alpha l = 90 \times \alpha = 0.1892\underline{/84°.32} \text{ hyp}$$
$$\sinh\theta = 0.1879\underline{/84°.39}$$
$$\cosh\theta = 0.982\underline{/0°.21}$$
$$A = D = \cosh\theta = 0.982\underline{/0°.21}$$
$$B = Z_0 \sinh\theta = 77.5\underline{/79°.84}$$
$$C = \frac{\sinh\theta}{Z_0} = 0.000456\underline{/88°.94}$$

Receiving transformers:

Rating = 75,000 kv.-a. (per circuit)
R = 0.5 per cent
X = 12 per cent
Iron loss = 0.6 per cent
Copper loss = 0.5 per cent
Magnetizing current = 5 per cent

$$I = \frac{75{,}000}{\sqrt{3} \times 220} = 196.9 \text{ amp.}$$

$$R_e = \frac{220{,}000 \times 0.005}{\sqrt{3} \times 196.9} = 3.223 \text{ ohms}$$

$$X_e = \frac{220{,}000 \times 0.12}{\sqrt{3} \times 196.9} = 77.4 \text{ ohms}$$

$$Z_r = 3.223 + j77.4 = 77.5\underline{/87°.61} \text{ ohms}$$

$$G_r = \frac{75{,}000 \times 1{,}000 \times 0.006}{3\left(\dfrac{220{,}000}{\sqrt{3}}\right)^2} = \frac{75 \times 0.6}{2.2^2} \times 10^{-6} = 9.30 \times 10^{-6} \text{ mho}$$

$$B_r = \frac{196.9 \times 0.05}{220{,}000/\sqrt{3}} = \frac{\sqrt{3} \times 196.9 \times 0.5}{2.2} 10^{-6} = 77.5 \times 10^{-6} \text{ mho}$$

$$Y_r = (0.93 - j7.75)10^{-5} = 7.80 \times 10^{-5}\overline{\backslash 83°.15} \text{ mho}$$

General circuit constants:

Equations (75) will be used.

$$\begin{aligned}
1 + \frac{Z_r Y_r}{2} &= 1 + \frac{77.5\underline{/87°.61} \times 7.80 \times 10^{-5}\overline{\backslash 83°.15}}{2} \\
&= 1 + 302.2 \times 10^{-5}\underline{/4°.46} = 1 + 0.00302 + j0.000235 \\
&= 1.00302 + j0.000235 = 1.00302\underline{/0°.01}
\end{aligned}$$

$$1 + \frac{Z_r Y_r}{4} = 1.00151 + j0.000118 = 1.00151\underline{/0°}$$

$$\begin{aligned}
A_0 &= 0.982\underline{/0°.21} \times 1.00302\underline{/0°.01} + 77.5\underline{/79°.84} \times 7.80 \times 10^{-5}\overline{\backslash 83°.15} \\
&= 0.985\underline{/0°.22} + 0.00605\overline{\backslash 3°.31} \\
&= 0.985 + j0.00378 + 0.00604 - j0.000350 \\
&= 0.99104 + j0.00343
\end{aligned}$$

$$\begin{aligned}
B_0 &= 77.5\underline{/79°.84} \times 1.00302\underline{/0°.01} + 0.982\underline{/0°.21} \times 77.5\underline{/87°.61} \times 1.00151\underline{/0°} \\
&= 77.75\underline{/79°.85} + 76.2\underline{/87°.82} \\
&= 13.71 + j76.5 + 2.895 + j76.1 \\
&= 16.605 + j152.6
\end{aligned}$$

$$\begin{aligned}
C_0 &= 0.000456\underline{/88°.94} \times 1.00302\underline{/0°.01} + 0.982\underline{/0°.21} \times 7.80 \times 10^{-5}\overline{\backslash 83°.15} \\
&= 45.75 \times 10^{-5}\underline{/88°.95} + 7.65 \times 10^{-5}\overline{\backslash 82°.94} \\
&= (0.837 + j45.75 + 0.940 - j7.59)10^{-5} \\
&= (1.777 + j38.16)10^{-5}
\end{aligned}$$

$$\begin{aligned}
D_0 &= 0.982\underline{/0°.21} \times 1.00302\underline{/0°.01} + 0.000456\underline{/88°.94} \times 77.5\underline{/87°.61} \times 1.00151\underline{/0°} \\
&= 0.985\underline{/0°.22} + 0.0354\underline{/176°.55} \\
&= 0.985 + j0.00378 - 0.0353 + j0.00213 \\
&= 0.9497 + j0.00591
\end{aligned}$$

EXAMPLE 2

Statement of Problem

The transmission line in Fig. 148 consists of two three-phase circuits with sectionalizing stations at points B and C, $\frac{1}{3}$ and $\frac{2}{3}$ of the transmission distance from the sending end. The banks of sending and receiving transformers are identical and have Z ohms short-circuit impedance and Y mhos excitation admittance per phase. Each section of the line has a surge impedance of Z_0 ohms and of hyperbolic angle of θ radians per circuit per phase.

Work out formal expressions for the general circuit constants for the system between the low-tension bus of the sending transformers and the low-

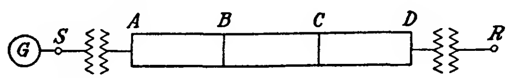

FIG. 148.—Single-line diagram of a two-circuit three-phase sectionalized transmission line with sending and receiving transformers. The sectionalizing stations are located at points B and C, one-third and two-thirds of the transmission distance from the sending end (Example 2).

tension bus of the receiving transformers when the line is operating with one circuit of section CD disconnected.

Solution

The necessary combination of circuit constants is carried through in Table XI. The general circuit constants for each of the five series sections of the network are first written down. Next, the general circuit constants of the transmission line alone are calculated by means of equations (46) for three networks in series. Finally, the formal expressions for the general circuit constants of the entire system have been obtained by applying equations (76) for a transmission line with sending and receiving transformers.

TABLE XI

General circuit constants	Sending transformers	Section 1	Section 2	Section 3	Receiving transformers
A	$1+\frac{ZY}{2}$	$\cosh\theta$	$\cosh\theta$	$\cosh\theta$	$1+\frac{ZY}{2}$
B	$Z\left(1+\frac{ZY}{4}\right)$	$\frac{Z_0\sinh\theta}{2}$	$\frac{Z_0\sinh\theta}{2}$	$Z_0\sinh\theta$	$Z\left(1+\frac{ZY}{4}\right)$
C	Y	$\frac{2\sinh\theta}{Z_0}$	$\frac{2\sinh\theta}{Z_0}$	$\frac{\sinh\theta}{Z_0}$	Y
D	$1+\frac{ZY}{2}$	$\cosh\theta$	$\cosh\theta$	$\cosh\theta$	$1+\frac{ZY}{2}$
A	$1+\frac{ZY}{2}$	$\cosh\theta(\cosh^2\theta+2\sinh^2\theta)$			$1+\frac{ZY}{2}$
B	$Z\left(1+\frac{ZY}{4}\right)$	$Z_0\sinh\theta(\sinh^2\theta+2\cosh^2\theta)$			$Z\left(1+\frac{ZY}{4}\right)$
C	Y	$\frac{\sinh\theta}{Z_0}(\sinh^2\theta+5\cosh^2\theta)$			Y
D	$1+\frac{ZY}{2}$	$\cosh\theta(\cosh^2\theta+5\sinh^2\theta)$			$1+\frac{ZY}{2}$
A_0	$\left(1+\frac{ZY}{2}\right)\left[\left(1+\frac{ZY}{2}\right)\cosh\theta(\cosh^2\theta+2\sinh^2\theta)+YZ_0\sinh\theta(\sinh^2\theta+2\cosh^2\theta)\right]+Z\left(1+\frac{ZY}{4}\right)\left[\left(1+\frac{ZY}{2}\right)\frac{\sinh\theta}{Z_0}(\sinh^2\theta+5\cosh^2\theta)+Y\cosh\theta(\cosh^2\theta+5\sinh^2\theta)\right]$				
B_0	$\left(1+\frac{ZY}{2}\right)\left[Z\left(1+\frac{ZY}{4}\right)\cosh\theta(\cosh^2\theta+2\sinh^2\theta)+\left(1+\frac{ZY}{2}\right)Z_0\sinh\theta(\sinh^2\theta+2\cosh^2\theta)\right]+Z\left(1+\frac{ZY}{4}\right)\left[Z\left(1+\frac{ZY}{4}\right)\frac{\sinh\theta}{Z_0}(\sinh^2\theta+5\cosh^2\theta)+\left(1+\frac{ZY}{2}\right)\cosh\theta(\cosh^2\theta+5\sinh^2\theta)\right]$				
C_0	$Y\left(1+\frac{ZY}{2}\right)\cosh\theta(\cosh^2\theta+2\sinh^2\theta)+YZ_0\sinh\theta(\sinh^2\theta+2\cosh^2\theta)\Big]+\left(1+\frac{ZY}{2}\right)\left[\left(1+\frac{ZY}{2}\right)\frac{\sinh\theta}{Z_0}(\sinh^2\theta+5\cosh^2\theta)+Y\cosh\theta(\cosh^2\theta+5\sinh^2\theta)\right]$				
D_0	$Y\left[Z\left(1+\frac{ZY}{4}\right)\cosh\theta(\cosh^2\theta+2\sinh^2\theta)+\left(1+\frac{ZY}{2}\right)Z_0\sinh\theta(\sinh^2\theta+2\cosh^2\theta)\right]+\left(1+\frac{ZY}{2}\right)\left[Z\left(1+\frac{ZY}{4}\right)\frac{\sinh\theta}{Z_0}(\sinh^2\theta+5\cosh^2\theta)+\left(1+\frac{ZY}{2}\right)\cosh\theta(\cosh^2\theta+5\sinh^2\theta)\right]$				

CHAPTER X

TRANSMISSION-LINE CHARTS

In modern transmission engineering, the analytical methods of solution no longer suffice. In studying the electrical characteristics of a projected transmission line preliminary to the actual design, or in analyzing the behavior of an interconnected transmission system, so many results are usually required that analytical solutions become prohibitive on account of the excessive amount of time and labor involved. Furthermore, many of the problems encountered today are so complex that analytical methods often fail because they prove fundamentally inadequate and hence incapable of handling the situation. In such cases, it becomes necessary to resort to graphical methods. The graphical solutions may be made as accurate as desired. If the diagrams and charts involved are constructed and used with care, a precision of results fully sufficient for engineering purposes may easily be obtained.

Without exaggeration it may be said that graphical methods constitute one of the electrical engineer's most important and powerful tools for handling many of the problems which he is called upon to solve today. No doubt, however, the importance of graphical methods will increase as the art progresses. It is but reasonable to expect that graphical methods, developed to meet the new demands, will play an even greater part in solving the still more advanced problems which are bound to confront the profession in the near future.

Whenever the relations between the electrical quantities at the terminals of a network or a system are expressible in terms of general circuit constants, its performance may also be determined graphically by a *circle diagram.* There are several such diagrams available. It is interesting to note that, fundamentally, they are all modifications of the well-known vector diagram which readily can be drawn for the electrical quantities at the sending and receiving end and have, therefore, been known for a long time, at least in principle. In the following will be discussed:

1. The circle diagram directly based on the vector diagram.
2. The Evans and Sels chart.
3. The modified Evans and Sels chart.

The last-mentioned one is probably the most convenient chart for use in transmission problems.

Circle Diagram Based Directly on Vector Diagram.[1]—Let the relations between sending and receiving voltages and currents of the network for which a circle diagram is desired be given by

$$E_s = A_0E_r + B_0I_r \tag{1}$$
$$I_s = C_0E_r + D_0I_r \tag{2}$$
$$E_r = D_0E_s - B_0I_s \tag{3}$$
$$I_r = -C_0E_s + A_0I_s \tag{4}$$

The general circuit constants may conveniently be expressed in polar form, as follows:

$$\left.\begin{aligned} A_0 &= |A_0|\underline{/\alpha} \\ B_0 &= |B_0|\underline{/\beta} \\ C_0 &= |C_0|\underline{/\gamma} \\ D_0 &= |D_0|\underline{/\delta} \end{aligned}\right\} \tag{5}$$

Figure 149 shows the vector diagram corresponding to equations (1) and (2), using the receiver voltage E_r as standard phase. The no-load sending voltage A_0E_r is displaced by an angle α from the receiver voltage, α being equal to the angle of the general circuit constant A_0. Splitting the receiving-end current I_r into an in-phase and a quadrature component, I_{rp} and I_{rq}, respectively,

[1] A circle diagram of this type for the *transmission line alone* is discussed in the paper "A Graphic Method for the Exact Solution of Transmission Lines," by C. H. Holladay, Trans. *A.I.E.E.*, p. 785, 1922.

A much more complete circle diagram of this type is described by L. Thielemans in a series of articles "Calculs, diagrammes et régulation des lignes de transport d'énergie à longue distance," *Revue Générale de l'Electricité*, Vol. VII, pp. 403, 435, 475, 515, 1920; Vol. IX, pp. 451, 599, 675, 878, 929, 1921. Thielemans' chart assumes constant voltage at one end of the system. It contains loci of the voltage at the other end, and of current, power, and power factor at both ends. In addition, it also gives the efficiency of transmission.

Space prevents a detailed discussion of the Thielemans chart in this treatise. Suffice it to say that it is an excellent chart, suitable for a variety of transmission problems. The author believes, however, that the modified Evans and Sels chart, which is described in this chapter, is more universally applicable and somewhat superior to the Thielemans chart for most purposes.

the component drops B_0I_{rp} and B_0I_{rq} may be laid off as shown. Evidently, the direction of the drop B_0I_{rp} due to the in-phase current is determined by the angle β of the general circuit constant B_0, and the direction of the drop B_0I_{rq} is 90 deg. displaced from the former or from the *unity power-factor line.* The drop B_0I_r, due to the total receiver current, is displaced an angle ϕ_r, the receiver power-factor angle, from the unity power-factor line. Having laid off the drops, the sending voltage vector is located as indicated in the figure.

Now assume that the receiver voltage E_r is constant. This immediately fixes the size and direction of A_0E_r and hence the

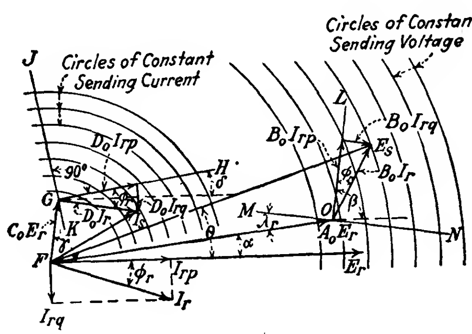

FIG. 149.—Receiver circle diagram based directly on vector diagram. This diagram is valid at one definite value of receiver voltage and may be used to determine the sending voltage, sending current, sending power factor, and, hence, also, the sending active and reactive power corresponding to any load condition at the receiver end.

point O. Along the lines OL and MN three scales may be constructed:

1. Scale of in-phase and quadrature current, respectively, in amperes.

2. Scale of voltage drop due to in-phase and quadrature current, respectively, in volts.

3. Scale of active and reactive power, respectively, at the receiver, in kilowatts and kilovolt-amperes.

The possibility of establishing also the power scale is obvious since power is voltage times current, and the receiver voltage is constant. As a matter of fact, the power scales are the important ones and the others are usually omitted. Using the power scales, the terminus of the sending voltage or *operating point* corre-

sponding to any value of active and reactive power received (or to active power and power factor) may immediately be located. Thus, the sending voltage for any operating condition at the receiving end can be determined. The loci of constant sending voltage are circles with the point F as center. A sufficient number of these circles may be drawn so that the values of the sending voltage can be easily read.

The sending current can also be determined graphically. C_0E_r is the no-load sending current. This vector is laid off from the point F (or if desired in a separate diagram) its size being fixed by the receiver voltage and its direction by the angle γ, the angle of the general circuit constant C_0. The line GH, giving the direction of the part of the sending current D_0I_{rp}, due to the in-phase component of the load current, is located by the angle δ, the angle of the general circuit constant D_0. The sending component D_0I_{rq} is displaced 90 deg. from GH and the sending current D_0I_r, due to the total load current, is displaced the receiver power-factor angle ϕ_r from this same line. The loci of constant sending current are circles about the point F, and may be drawn in.

In order rapidly to determine the sending current corresponding to given operating conditions at the receiver end as specified by active and reactive power received, a scale properly reduced but still reading kilowatts and reactive kilovolt-amperes received may be laid off along the lines GH and JK, respectively. The terminus of the sending-current vector can then immediately be located when the receiver conditions are known. The active and reactive power at the sending end may now be calculated by

$$P_s = E_sI_s \cos \phi_s \tag{6}$$

$$Q_s = E_sI_s \sin \phi_s \tag{7}$$

where ϕ_s is the sending power-factor angle which is scaled from the diagram. The accuracy of the sending power determined in this manner depends mainly on the precision with which the angle ϕ_s can be read.

The scales of voltage, current, and power are interdependent. In constructing the diagram, one scale can be fixed arbitrarily; the others, however, must be calculated on the basis of the chosen scale. Usually it is most convenient to select the power scale, particularly if coordinated paper is used. The following equations give the relation between the scales

$$\left.\begin{aligned} \text{Power scale} &= p \text{ watts per inch} \\ \text{Voltage scale} &= \frac{pB_0}{E_r} \text{ volts per inch} \\ \text{Current scale} &= \frac{p}{E_r} \text{ amp. per inch} \end{aligned}\right\} \quad (8)$$

If coordinated paper is used for the diagram MN will be located along one of the coordinate lines and the point O chosen. It is convenient then to locate the point F by determining the direction of the vector A_0E_r relative to the line MN by means of the angle λ_r which is given by

$$\lambda_r = 90 + \alpha - \beta \quad (9)$$

The circle diagram so far discussed was based on equations (1) and (2). In identically the same way, a circle diagram may be constructed on the basis of equations (3) and (4). This diagram will have the sending voltage E_s as reference vector and will give the receiver voltage corresponding to any value of active and reactive power at the sending end. It may also give the receiver current and power-factor angle, so that the active and reactive power at the receiver end may be calculated by

$$P_r = E_rI_r \cos \phi_r \quad (10)$$
$$Q_r = E_rI_r \sin \phi_r \quad (11)$$

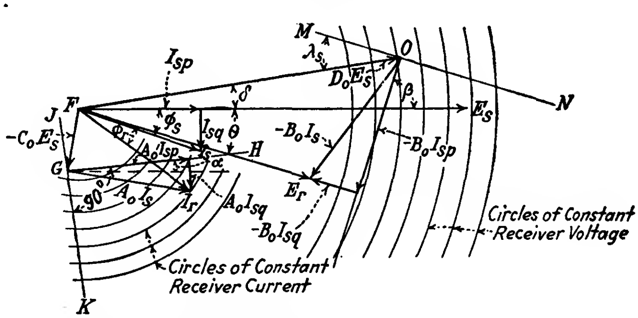

FIG. 150.—Sending circle diagram based directly on vector diagram. This diagram is valid at one definite value of sending voltage and may be used to determine the receiving voltage, receiving current, receiving power factor, and, hence, also, the receiving active and reactive power corresponding to any condition of power and reactive power at the sending end.

Figure 150 shows this diagram which, presumably, does not need a detailed description, since the principles involved are exactly the same as in the preceding one.

The scales are determined by

$$\left.\begin{aligned} \text{Power scale} &= p \text{ watts per inch} \\ \text{Voltage scale} &= \frac{pB_0}{E_s} \text{ volts per inch} \\ \text{Current scale} &= \frac{p}{E_s} \text{ amp. per inch} \end{aligned}\right\} \quad (12)$$

The angle λ_s, which may conveniently be made use of in constructing the diagram, is given by

$$\lambda_s = 90 + \delta - \beta \qquad (13)$$

If an operating point has been located in, for instance, the receiver diagram (Fig. 149), then the corresponding sending operating point may be found in the sending diagram (Fig. 150), *provided the latter is constructed for exactly the sending voltage necessary to produce the given receiver conditions.* If this is the case, the operating point in the sending chart can be determined as follows: First, this operating point must lie on the circle representing the proper receiver voltage. Second, the angular displacement between the sending and receiving voltage must be the same in the two diagrams. Hence, this angle θ may be read in the receiving chart and used in the sending chart so that the direction of the receiver voltage is determined. Since now the receiver voltage is known, both in magnitude and direction, the sending operating point is immediately located and the active and reactive power determined. The same scheme may evidently also be used in transferring a point from the sending to the receiving chart, but its application in this case is, of course, subject to the same limitation as in the former.

It is possible, however, by introducing variable scales to make these diagrams universal, *i.e.*, applicable without the strictly imposed condition that one of the voltages must be fixed. This will not be discussed here but will be taken up in connection with the Evans and Sels chart.

In their present shape, the diagrams so far discussed are not altogether suitable where power conditions at both ends of the system are required, and particularly not where corresponding values of sending and receiver power have to be determined at a variety of terminal voltages. There are problems, however, for which these diagrams are entirely satisfactory. It all depends upon what the conditions are and what results are desired.

EXAMPLE 1

This example illustrates the use of the receiver diagram just described in the solution of a simple transmission problem. It will be noted that all lines in the diagram superfluous for this particular problem have been omitted.

Statement of Problem

A three-phase, 60-cycle transmission line, 100 miles long, with 9-ft. spacing between solid conductors of No. 000 wire, has the following constants per wire mile:

$$\left.\begin{aligned} r &= 0.342 \text{ ohm}\ldots\ldots\ldots\ldots \\ x &= 0.788 \text{ ohm}\ldots\ldots\ldots\ldots \\ y &= j5.38 \times 10^{-6} \text{ mho}\ldots\ldots \end{aligned}\right\} \text{From Westinghouse Tables[1] (at 25°C.)}$$

The voltage at the load is to be maintained = 66,000 volts between lines.

1. Obtaining data from a simple graphical solution based on the nominal Π line, plot sending-end voltage versus received load for 100, 90, 80, and 70 per cent power factor (lagging). Let the total load vary from 0 to 10,000 kw.

2. Determine the per cent regulation for a load of 4,500 kv.-a. at 80 per cent power factor.

Solution

1. The constants of the nominal Π are

$$Z = 34.2 + j78.8 = 86.0\underline{/66°.5} \text{ ohms}$$

$$Y = j5.38 \times 10^{-4} = 5.38 \times 10^{-4}\underline{/90°} \text{ mhos}$$

The relation between the receiving-end and sending-end voltage is given by

$$V_s = V_r\left(1 + \frac{YZ}{2}\right) + I_B Z = A_0 V_r + B_0 I_r \qquad (a)$$

$$A_0 = 1 + \frac{YZ}{2} = 1 + \frac{5.38 \times 10^{-4}\underline{/90°} \times 86.0\underline{/66°.5}}{2}$$

$$= 0.9788 + j0.00923 = 0.979\underline{/0°.5}$$

Scales for the graphical solution

$$1 \text{ in.} = 2{,}000 \text{ kw. (total)}$$

$$\text{Per phase}\begin{cases} p = \dfrac{2{,}000 \times 10^3}{3} = \dfrac{2}{3} \times 10^6 \text{ watts per inch} \\[2ex] v = \dfrac{PZ}{V_r} = \dfrac{2 \times 10^6 \times 86.0 \times \sqrt{3}}{3 \times 66{,}000} = 1{,}505 \text{ volts per inch} \end{cases}$$

$$\left(1 + \frac{YZ}{2}\right)V_r = \frac{0.979 \times 66{,}000}{\sqrt{3} \times 1{,}505} = 24.78 \text{ in.}$$

$$\text{Angle } \lambda_r = 90 + 0.5 - 66.5 = 24.0 \text{ deg.}$$

Total power and line voltage are indicated on the graphical solution of Fig. 151 from which the following values are obtained:

[1] These tables will be found in "Electrical Characteristics of Transmission Circuits," published by the Westinghouse Electric and Manufacturing Company, as Reprint 82, February, 1922.

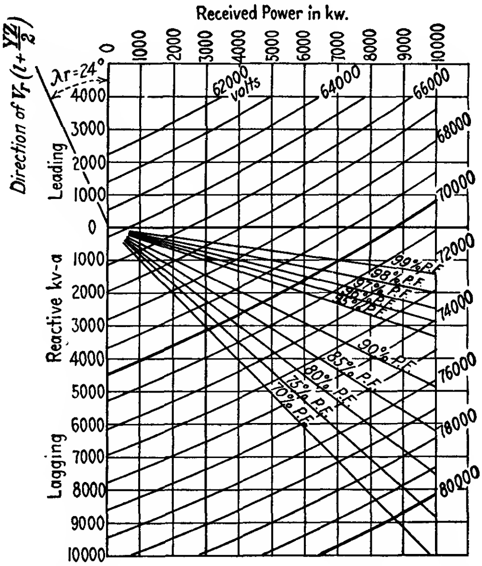

FIG. 151.—Receiver circle diagram for the 100-mile transmission line considered in Example 1. The solution is based on the nominal Π. Receiver voltage equal to 66,000 volts between lines.

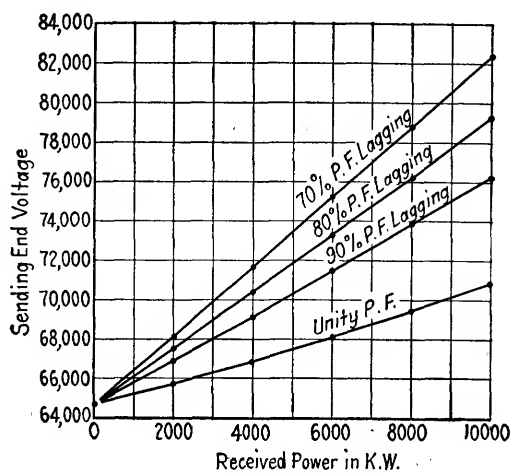

FIG. 152.—100-mile transmission line (Example 2). Receiver voltage equal to 66,000 volts. The curves give sending voltage versus power received for values of power factors ranging from 100 per cent to 70 per cent lagging.

Total load, kilowatts	Sending-end voltage (volts between lines)			
	100 per cent power factor	90 per cent power factor	80 per cent power factor	70 per cent power factor
0	64,650	64,650	64,650	64,650
2,000	65,700	66,850	67,450	68,100
4,000	66,850	69,150	70,400	71,600
6,000	68,100	71,500	73,300	75,200
8,000	69,450	73,900	76,250	78,750
10,000	70,850	76,250	79,300	82,400

Figure 152 shows the curves of sending voltage *versus* received power for the desired values of power-factor.

2. Regulation for a load of 4,500 kv.-a. at 80 per cent power-factor.

Sending-end voltage = 69,850 volts (obtained from the 80 per cent power-factor curve at 3,600 kw.)

When the load is removed, the receiver voltage rises to

$$V_r' = \frac{V_s}{1 + \frac{YZ}{2}} = \frac{69{,}850}{0.979} = 71{,}350 \text{ volts}$$

$$\text{Regulation} = \frac{71{,}350 - 66{,}000}{66{,}000} 100 = 8.10 \text{ per cent}$$

The Evans and Sels Chart.—This chart was presented by Evans and Sels in 1921.[1] Although, at first glance, it may appear to differ materially from those previously described, it is fundamentally a vector diagram. It embodies, however, certain added features such as loci of constant line loss and constant efficiency of transmission.

Voltage Circles.—Let the general equations for the system be

$$E_s = A_0E_r + B_0I_r \tag{14}$$
$$I_s = C_0E_r + D_0I_r \tag{15}$$
$$E_r = D_0E_s - B_0I_s \tag{16}$$
$$I_r = -C_0E_s + A_0I_s \tag{17}$$

[1] Evans, R. D., and H. K. Sels, "Circle Diagrams for Transmission Systems," *Elec. Jour.*, p. 530, December, 1921. In this article the calculations of the data necessary for the construction of the charts were systematized in an excellent manner. All chart constants were given in terms of the general circuit constants.

A voltage-power circle diagram of this type is also described in H. B. Dwight's "Constant-voltage Transmission," John Wiley Sons, Inc., New York, 1915.

Also, let the general circuit constants be given in polar and rectangular form as follows

$$
\begin{aligned}
A_0 &= |A_0|\underline{/\alpha} = a_1 + ja_2 \\
B_0 &= |B_0|\underline{/\beta} = b_1 + jb_2 \\
C_0 &= |C_0|\underline{/\gamma} = c_1 + jc_2 \\
D_0 &= |D_0|\underline{/\delta} = d_1 + jd_2
\end{aligned} \tag{18}
$$

Equation (14) may then be written

$$E_s = (a_1 + ja_2)E_r + (b_1 + jb_2)I_r \tag{19}$$

Using the receiver voltage E_r as standard phase and introducing the in-phase and quadrature components of the receiver current I_r, equation (19) becomes

$$E_s = (a_1 + ja_2)E_r + (b_1 + jb_2)(I_{rp} + jI_{rq}) \tag{20}$$

Multiplying out and collecting in-phase and quadrature terms gives

$$E_s = a_1E_r + b_1I_{rp} - b_2I_{rq} + j(a_2E_r + b_2I_{rp} + b_1I_{rq}) \tag{21}$$

This is a vector equation which, upon squaring, turns into the following algebraic equation:

$$E_s^2 = (a_1^2 + a_2^2)E_r^2 + 2(a_1b_1 + a_2b_2)E_rI_{rp} + 2(a_2b_1 - a_1b_2)E_rI_{rq} + (b_1^2 + b_2^2)I_{rp}^2 + (b_1^2 + b_2^2)I_{rq}^2 \tag{22}$$

which may be arranged as follows:

$$I_{rp}^2 + \frac{2(a_1b_1 - a_2b_2)E_r}{b_1^2 + b_2^2}I_{rp} + I_{rq}^2 + \frac{2(a_2b_1 - a_1b_2)E_r}{b_1^2 + b_2^2}I_{rq} = \frac{E_s^2 - (a_1^2 + a_2^2)E_r^2}{b_1^2 + b_2^2} \tag{23}$$

The squares on the left-hand side may be completed by the addition of

$$\left(\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}\right)^2 E_r^2 + \left(\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}\right)^2 E_r^2 = \frac{(a_1b_1 + a_2b_2)^2 + (a_2b_1 - a_1b_2)^2}{(b_1^2 + b_2^2)^2}E_r^2 \tag{24}$$

The result is

$$\left[I_{rp} + \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}E_r\right]^2 + \left[I_{rq} + \frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}E_r\right]^2 = \frac{E_s^2}{b_1^2 + b_2^2} \tag{25}$$

If I_{rp} and I_{rq} are taken as variables, equation (25) obviously represents the equation of a circle. The position of the center is given by its displacements from the coordinate axes, as follows:

$$\text{Horizontal displacement} = -\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}E_r \tag{26}$$

$$\text{Vertical displacement} = -\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}E_r \tag{27}$$

and the radius of the circle is given by

$$\text{Radius} = \frac{E_s}{\sqrt{b_1^2 + b_2^2}} \tag{28}$$

A circle diagram based on equation (25) would be a voltage-current diagram and might be useful for certain purposes. As a rule, however, a voltage-power diagram is required and the equations for this will now be established. Further comment on the current diagram at this point seems unnecessary since the general discussion of the power diagram which will follow is directly applicable also to the current diagram.

Equation (25) may be converted into a power equation by introducing

$$I_{rp} = \frac{P_r}{E_r} \text{ and } I_{rq} = \frac{Q_r}{E_r} \tag{29}$$

The result is

$$\left[P_r + \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}E_r^2\right]^2 + \left[Q_r + \frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}E_r^2\right]^2 = \frac{E_s^2E_r^2}{b_1^2 + b_2^2} \tag{30}$$

In a polyphase system P_r and Q_r are the power (in watts) and reactive power (in volt-amperes), respectively, *per phase*, and the voltages are *to neutral.* Since, however, the three-phase system is paramount in importance, the formulas will be especially adapted to this system and modified to read in terms of *total kilowatts and reactive kilovolt-amperes* in connection *with line voltage in volts.* Capital subscripts will designate total power in kilowatts or kilovolt-amperes and line voltage in volts. Thus,

$$P_r = \frac{1{,}000P_R}{3} \text{ and } Q_r = \frac{1{,}000Q_R}{3} \tag{31}$$

$$E_s = \frac{E_S}{\sqrt{3}} \text{ and } E_r = \frac{E_R}{\sqrt{3}} \tag{32}$$

Inserting equations (31) and (32) in equation (30) gives

$$\left[P_R + \frac{1}{1{,}000}\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}E_R^2\right]^2 + \left[Q_R + \frac{1}{1{,}000}\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}E_R^2\right]^2 = \left(\frac{1}{1{,}000}\right)^2\frac{E_S^2E_R^2}{b_1^2 + b_2^2} \quad (33)$$

which may be written in simpler terms, as

$$\left[P_R + \frac{l}{1{,}000}E_R^2\right]^2 + \left[Q_R + \frac{m}{1{,}000}E^2\right]^2 = \left(\frac{n}{1{,}000}\right)^2 E_S^2E_R^2 \quad (34)$$

or,

$$[P_R - A_R]^2 + [Q_R - B_R]^2 = C_R^2 \quad (35)$$

The last equation is the well-known standard expression for a circle with center in the first quadrant assuming A_R and B_R positive, and radius C_R. Hence, the displacement of the center and the radius are given by

$$A_R = -\frac{l}{1{,}000}E_R^2 = -\frac{1}{1{,}000}\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}E_R^2 \quad (36)$$

$$B_R = -\frac{m}{1{,}000}E_R^2 = -\frac{1}{1{,}000}\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}E_R^2 \quad (37)$$

$$C_R = \frac{n}{1{,}000}E_SE_R = \frac{1}{1{,}000}\frac{E_SE_R}{\sqrt{b_1^2 + b_2}} \quad (38)$$

Since A_R is negative, the center will be displaced horizontally in a negative direction (to the left). In a system with ordinary constants the product a_1b_2 will be much greater than a_2b_1. B_R will consequently come out positive and the center will be displaced vertically in a positive direction (upward). In other words, the center will lie in the second quadrant.

If the receiver voltage E_R is constant, the position of the center is fixed. A family of concentric circles may then be drawn about this center, each of which will represent one definite value of sending voltage. Figure 153 shows such a diagram. This chart, *the receiving chart for constant receiver voltage,* will immediately give the sending voltage corresponding to any value of active and reactive power at the receiver end, the receiving voltage being fixed.

If the sending voltage is constant and the receiver voltage variable, another chart, *the receiving chart for constant sending voltage,* may be constructed from equation (34). In this the

position of the center evidently will not be fixed, but will depend on the chosen values of receiving voltage. The centers will,

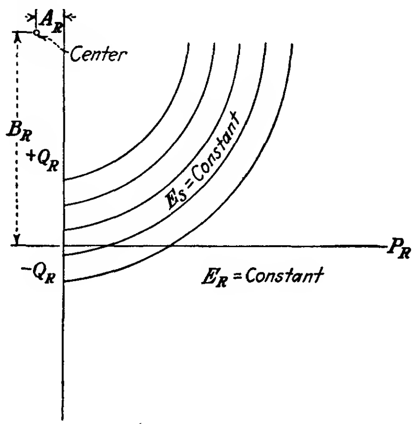

FIG. 153.—Evans and Sels receiver chart for constant receiver voltage. Each of the concentric circles represents constant sending voltage. When the active and reactive power received are known, therefore, the sending voltage can be determined.

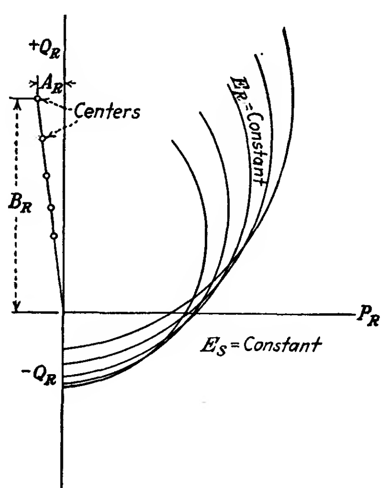

FIG. 154.—Evans and Sels receiver chart for constant sending voltage. Each of the eccentric circles represents constant receiver voltage. If the receiver power and reactive power, therefore, are known, the receiving voltage can be determined.

however, all lie on a straight line passing through the origin. A family of eccentric circles each representing a definite value of the

receiver voltage may be drawn about the centers. This chart is shown in Fig. 154. It gives the receiver voltage corresponding to any value of active and reactive power at the receiving end, the sending voltage being fixed.

The charts so far discussed involve receiver power only. By basing the derivation of the circle equation on equation (16), diagrams having active and reactive power at the sending end as coordinates may also be constructed. Thus equation (16) may be written

$$E_r = (d_1 + jd_2)E_s - (b_1 + jb_2)(I_{sp} + jI_{sq}) \tag{39}$$

in which the sending voltage is used as reference vector. Since this equation is entirely similar to equation (20), the only difference being the minus sign, all the intermediate steps necessary in the preceding derivation may now be omitted and the final voltage-current equation written down immediately by help of equation (25). The result is

$$\left[I_{sp} - \frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2}E_s\right]^2 + \left[I_{sq} - \frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2}E_s\right]^2 = \frac{E_r^2}{b_1^2 + b_2^2} \tag{40}$$

The position of the center is determined as follows

$$\text{Horizontal displacement} = \frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2} \tag{41}$$

$$\text{Vertical displacement} = \frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2} \tag{42}$$

and the radius of the circle is given by

$$\text{Radius} = \frac{E_r}{\sqrt{b_1^2 + b_2^2}} \tag{43}$$

Equation (41) is converted into a power equation by introducing

$$I_{sp} = \frac{P_s}{E_s} \text{ and } I_{sq} = \frac{Q_s}{E_s} \tag{44}$$

It becomes

$$\left[P_s - \frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2}E_s^2\right]^2 + \left[Q_s - \frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2}E_2\right]^2 = \frac{E^2E_r^2}{b_1^2 + b_2^2} \tag{45}$$

Again adapting this to a three-phase system using *total power in kilowatts, total reactive power in kilovolt-amperes, and line voltage in volts,* the result is

$$\left[P_S - \frac{1}{1{,}000}\,\frac{d_1b_1 + d_2b_2}{b^2 + b_2^2}E_S^2\right]^2 + \left[Q_S - \frac{1}{1{,}000}\,\frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2}E_S^2\right]^2 = \left(\frac{1}{1{,}000}\right)^2 \frac{E_S^2E_R^2}{b_1^2 + b_2^2} \tag{46}$$

which may be written

$$\left[P_S - \frac{l'}{1{,}000}E_S^2\right]^2 + \left[Q_S - \frac{m'}{1{,}000}E_S^2\right]^2 = \left(\frac{n'}{1{,}000}\right)^2 E_S E_R^2 \tag{47}$$

or, still simpler,

$$[P_S - A_S]^2 + [Q_S - B_S]^2 = C_S^2 \tag{48}$$

Hence, the displacement of the center and the radius are given by

$$A_S = \frac{l'}{1{,}000}\,E_S^2 = \frac{1}{1{,}000}\,\frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2}\,E_S^2 \tag{49}$$

$$B_S = \frac{m'}{1{,}000}\,E_S^2 = \frac{1}{1{,}000}\,\frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2}\,E_S^2 \tag{50}$$

$$C_S = \frac{n'}{1{,}000}\,E_SE_R = \frac{1}{1{,}000}\,\frac{E_SE_R}{\sqrt{b_1^2 + b_2^2}} \tag{51}$$

It should be noted that $n' = n$ and $C_S = C_R$. Since A_S is positive, the center will be displaced horizontally in a positive direction. B_S will always be negative in ordinary systems since d_1b_2 is greater than d_2b_1. Hence the center will be displaced vertically in a negative direction. In other words, the center will lie in the fourth quadrant.

If the sending voltage E_S is constant, a *sending chart for constant sending voltage* may be constructed, as shown in Fig. 155. This chart immediately gives the receiver voltage corresponding to any value of active and reactive power at the sending end, the sending voltage being fixed.

If the receiver voltage is constant and the sending voltage variable, a *sending chart for constant receiver voltage* may be drawn as indicated in Fig. 156. This chart gives the sending voltage corresponding to any value of active and reactive power at the sending end, the receiver voltage being fixed.

Angular Relations.—The angular displacement (phase angle) between the sending and receiving voltages may also be obtained from the preceding charts. Take the receiver chart for constant receiver voltage, for instance, and draw the line FO connecting

the center with the origin, as shown in Fig. 157. This line represents the no-load sending voltage. Also draw a line FR displaced

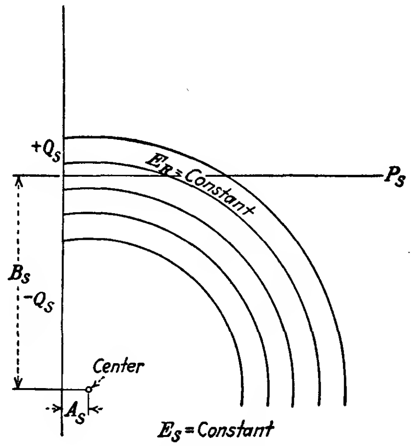

FIG. 155.—Evans and Sels sending chart for constant sending voltage. Each of the concentric circles represents constant receiver voltage. If the sending power and reactive power, therefore, are known, the receiver voltage can be determined.

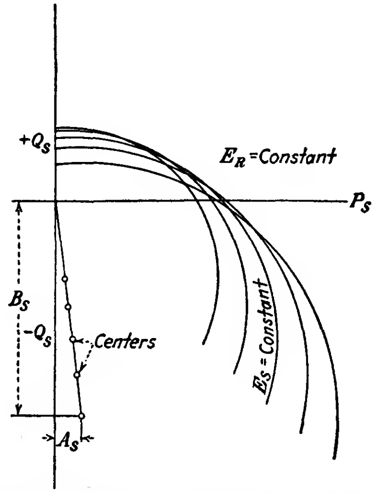

FIG. 156.—Evans and Sels sending chart for constant receiver voltage. Each of the eccentric circles represents constant sending voltage. If the sending, active, and reactive power, therefore, are known, the sending voltage can be determined.

by an angle α, the angle of A_0, from FO in the direction shown. The line FR then indicates the phase position of the receiver

voltage E_r with respect to the rest of the diagram. Any line from the center to a sending-voltage circle such as FS represents correctly a sending voltage in magnitude and phase and the phase displacement between the voltages at the two ends is directly given by the angle θ.

Without resorting to an analytical proof, the correctness of this statement may be demonstrated by comparing the diagram (Fig. 157) with the original vector diagram (Fig. 149) and showing that the two are fundamentally the same. The diagram (Fig. 157) may be superimposed on Fig. 149 by placing the origin O of

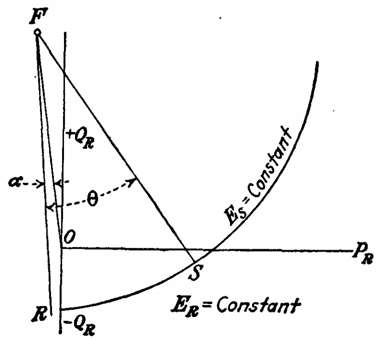

FIG. 157.—This figure shows how the displacement θ between the sending and receiving voltage can be determined in the Evans and Sels receiver chart for constant receiving voltage. The reference line (the direction of the receiver voltage) is located as indicated by means of the angle α, the angle of the general circuit constant A_0.

the former on point O in the latter, so that the power and reactive power axes of Fig. 157 coincide with OL and MN, respectively, in Fig. 149. Assuming identical power scales in the two diagrams, their *sending-voltage circles must, of necessity, coincide.* Hence, the circles must have the same center, and, consequently, the lines FO in the two diagrams, as well as the direction of the line FR in Fig. 157 and the direction of the receiver voltage vector E_r in Fig. 149, will coincide. This being the case, it immediately follows that the angular displacements between voltages may be found in the diagram (Fig. 157), as pointed out.

A family of angle lines, *i.e.*, straight lines radiating from the center, may be drawn in so that the phase angle between voltages may be read at a glance. Evidently the same may be done in the sending chart for constant sending voltage as shown in Fig. 158. In this case the reference line (the sending-voltage vector) is displaced the angle δ from OF in the direction shown. The correctness of this may be verified by a comparison with Fig. 150.

As a matter of fact, the angular relations may also be obtained from the receiver chart for constant sending voltage (Fig. 154) and from the sending chart for constant receiver voltage (Fig. 156). If a complete set of angle lines were to be drawn in these charts, however, they would become rather crowded, since, due to the eccentricity of the voltage circles, *each one would require its own family of angle lines emanating from its own center.* Hence, if it is desired to obtain phase angles from these charts, it may be more convenient merely to draw the reference lines (one for each center) and use a protractor to read the angles.

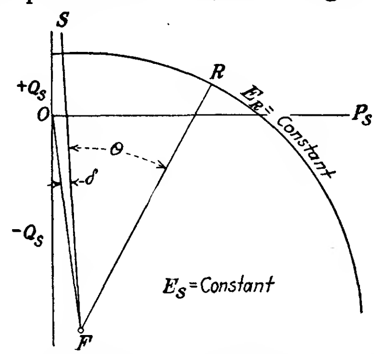

FIG. 158.—This figure shows how the displacement θ between the sending and receiving voltage can be determined in the Evans and Sels sending chart for constant sending voltage. The reference line (the direction of the sending voltage) is located as indicated by the angle δ, the angle of the general circuit constant D_0.

The phase displacements between the sending and receiving voltages when used in connection with their magnitudes make it possible to determine the conditions as regards active and reactive power at one end corresponding to the conditions existing at the other. In other words, they may be of assistance in transferring an operating point from one chart to another. Since, however, the use of the angular relations is rather inconvenient in the charts having eccentric voltage circles, it is usually more practical to make use of power loss for this purpose.

Loss Circles.—In the receiver chart for constant receiver voltage and in the sending chart for constant sending voltage, it is possible to locate loci of constant loss. These are concentric circles and can easily be drawn in.

The power lost in transmission is given by the difference between the sending and receiving power, *i.e.*,

$$\text{Loss} = P_s - P_r \text{ watts per phase} \tag{52}$$

In order to derive a general expression for the power loss in terms of the general circuit constants, it is convenient to express the sending power as the real part of the product of the sending voltage and the *conjugate* of the sending current, and, similarly, the receiver power as the real part of the product of the receiver voltage and the conjugate of the receiver current.[1] Designating the conjugate of a complex quantity by a dash above the symbol, the loss may be written

$$\text{Loss} = (E_s\bar{I}_s - E_r\bar{I}_r)_{\text{real}} \tag{53}$$

By substituting in this equation the sending voltage and current in terms of the receiver voltage and current, or *vice versa*, the power loss is obtained in terms of active and reactive power at the receiver end or sending end, respectively. Use the symbols L_r and L_s for the loss per phase in the two cases. By means of equation (53), in connection with equations (14) and (15), L_r may be written

$$\begin{aligned} L_r = {} & [(A_0E_r + B_0I_r)(\bar{C}_0\bar{E}_r + \bar{D}_0\bar{I}_r) - E_r\bar{I}_r]_{\text{real}} = \\ & [A_0\bar{C}_0E_r\bar{E}_r + B_0\bar{C}_0\bar{E}_rI_r + A_0\bar{D}_0E_r\bar{I}_r + B_0\bar{D}_0I_r\bar{I}_r - E_r\bar{I}_r]_{\text{real}} = \\ & [(a_1 + ja_2)(c_1 - jc_2)E_r^2 + (b_1 + jb_2)(c_1 - jc_2)E_r(I_{rp} + jI_{rq}) + \\ & (a_1 + ja_2)(d_1 - jd_2)E_r(I_{rp} - jI_{rq}) + (b_1 + jb_2)(d_1 - jd_2) \\ & \qquad (I_{rp}^2 + I_{rq}^2) - E_r(I_{rp} - jI_{rq})]_{\text{real}} \end{aligned} \tag{54}$$

Multiplying out and collecting the real terms gives

$$\begin{aligned} L_r = {} & (a_1c_1 + a_2c_2)E_r^2 + (b_1c_1 + b_2c_2 + a_1d_1 + a_2d_2 - 1)E_rI_{rp} - \\ & (b_2c_1 - b_1c_2 + a_1d_2 - a_2d_1)E_rI_{rq} + (d_1b_1 + d_2b_2)(I_{rp}^2 + I_{rq}^2) \\ = {} & uE_r^2 + tP_r - wQ_r + v\frac{P_r^2 + Q_r^2}{E_r^2} \end{aligned} \tag{55}$$

Rearranging as follows:

$$P_r^2 + \frac{t}{v}E_r^2P_r + Q_r^2 - \frac{w}{v}E_r^2Q_r = \frac{L_r}{v}E_r^2 - \frac{u}{v}E_r^4 \tag{56}$$

and adding

$$\left(\frac{t}{2v}\right)^2E_r^4 + \left(\frac{w}{2v}\right)^2E_r^4 = \frac{t^2 + w^2}{4v^2}E_r^4 \tag{57}$$

[1] It may be noted that the imaginary part of these products represent reactive power. When the latter quantity is desired, however, it is preferable to take the *product of the current and the conjugate of the voltage*, as this procedure will give the reactive power with its correct sign. It is assumed that leading reactive power is considered positive and lagging, negative. As far as the active power is concerned, it makes no difference which product is used. The real part will be the same, in either case.

in order to complete the squares, gives

$$\left[P_r + \frac{t}{2v}E_r^2\right]^2 + \left[Q_r - \frac{w}{2v}E_r^2\right]^2 = E_r^2\left[\frac{L_r}{v} + \frac{t^2 + w^2 - 4uv}{4v^2}E_r^2\right] \quad (58)$$

Adapting this equation specifically to a three-phase system, as was previously done for the voltage-power equations, results in

$$\left[P_R + \frac{t}{2{,}000v}E_R^2\right]^2 + \left[Q_R - \frac{w}{2{,}000v}E_R^2\right]^2 = \left(\frac{E_R}{1{,}000}\right)^2\left[\frac{L_R}{v} + \frac{t^2 + w^2 - 4uv}{4v^2}E_R^2\right] \quad (59)$$

Here L_R represents the total loss in watts. The equation represents, as seen, a circle whose center is fixed (assuming constant receiver voltage) and whose radius depends upon the value of the loss. The displacement of the center and the radius are given by

$$A_{RL} = -\frac{t}{2{,}000v}E_R^2 \quad (60)$$

$$B_{RL} = \frac{w}{2{,}000v}E_R^2 \quad (61)$$

$$C_{RL} = \frac{E_R}{1{,}000}\sqrt{\frac{L_R}{v} + \frac{t^2 + w^2 - 4uv}{4v^2}E_R^2} \quad (62)$$

The auxiliary constants calculated from the general circuit constants are

$$\begin{aligned} t &= b_1c_1 + b_2c_2 + a_1d_1 + a_2d_2 - 1 \\ &= 2(b_1c_1 + a_2d_2)^1 & (63) \\ u &= a_1c_1 + a_2c_2 & (64) \\ v &= d_1b_1 + d_2b_2 & (65) \\ w &= b_2c_1 - b_1c_2 + a_1d_2 - a_2d_1 & (66) \end{aligned}$$

Since t will be positive and w negative for ordinary systems, the center usually lies in the third quadrant. By assuming appropriate values of the loss, the radii of a family of concentric circles, each representing constant loss, may be computed and plotted as indicated in Fig. 159. This diagram, then, immediately gives the loss corresponding to any condition of active and reactive power at the receiver end, the receiver voltage being fixed.

Adding the receiver power and the loss determines the sending power. Sending power and voltage both being known definitely fixes the corresponding operating point in the sending chart.

[1] This reduction depends on the relation $A_0D_0 - B_0C_0 = 1$.

The power loss in terms of sending active and reactive power may be determined from equation (53) by substitution of equations (16) and (17). This gives

$$L_s = [E_s\bar{I}_s - (D_0E_s - B_0I_s)(-\bar{C}_0\bar{E}_s + \bar{A}_0\bar{I}_s)]_{\text{real}} \qquad (67)$$

By a process similar to the one previously employed in connection with equation (54), equation (67) may be brought into the following form:

$$\left[P_s - \frac{t}{2v'}E_s^2\right]^2 + \left[Q_s + \frac{w'}{2v'}E_s^2\right]^2 = E_s^2\left[\frac{L_s}{v'} + \frac{t^2 + (w')^2 - 4u'v'}{4(v')^2}E_s^2\right] \qquad (68)$$

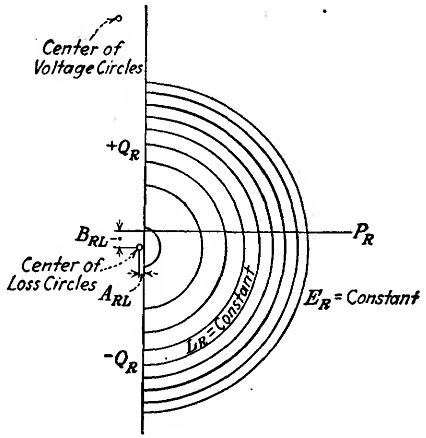

FIG. 159.—This figure shows how the loci of constant loss can be located in the Evans and Sels receiver chart for constant receiver voltage. These loci are concentric circles. For constant loss increment the spacing between the circles decreases as the loss increases. From this diagram the power lost in transmission can be determined when the active and reactive power at the receiving end are known.

which for a three-phase system may be written

$$\left[P_S - \frac{t}{2{,}000v'}E_S^2\right]^2 + \left[Q_S + \frac{w'}{2{,}000v'}E_S^2\right]^2 = \left(\frac{E_S}{1{,}000}\right)^2\left[\frac{L_S}{v'} + \frac{t^2 + (w')^2 - 4u'v'}{4(v')^2}E_S^2\right] \qquad (69)$$

The displacement of the center and the radius are given by

$$A_{SL} = \frac{t}{2{,}000\,v'}E_S^2 \qquad (70)$$

$$B_{SL} = -\frac{w'}{2{,}000\, v'} E_S^2 \tag{71}$$

$$C_{SL} = \frac{E_S}{1{,}000}\sqrt{\frac{L_S}{v'} + \frac{t^2 + (w')^2 - 4u'v'}{4(v')^2} E_S^2} \tag{72}$$

The additional auxiliary constants needed in this case are

$$u' = d_1c_1 + d_2c_2 \tag{73}$$

$$v' = a_1b_1 + a_2b_2 \tag{74}$$

$$w' = b_2c_1 - b_1c_2 + a_2d_1 - a_1d_2 \tag{75}$$

Figure 160 shows a family of loss circles in the sending chart for constant sending voltage. As seen, the center is located in the first quadrant. This will usually be the case, since w' (as well as w), as a rule, is negative.

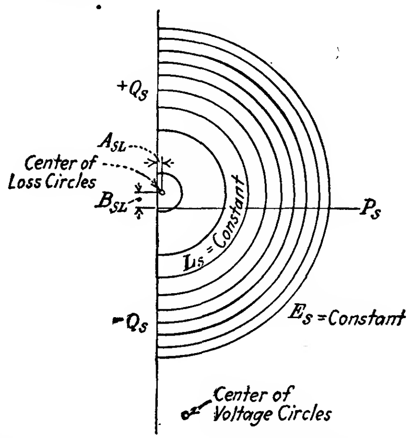

FIG. 160.—This figure shows how loci of constant loss can be located in the Evans and Sels sending chart for constant sending voltage. These loci are concentric circles. For constant loss increment the spacing between the circles decreases as the loss increases. From this diagram the power lost in transmission can be determined when the active and reactive power at the sending end are known.

If the system is symmetrical $u = u'$, $v = v'$, and $w = w'$, since $A_0 = D_0$. This would hold, for instance, in the case of a transmission line with identical terminal transformers or a transmission line alone. If, in the latter case, the leakance is negligible, c_1 is practically zero, and the auxiliary constants become

$$t = b_2c_2 + a_1^2 + a_2^2 - 1 = 2a_2d_2 \cong 0 \text{ (approximately)} \tag{76}$$

$$u = u' = a_2c_2 \tag{77}$$

$$v = v' = a_1b_1 + a_2b_2 \tag{78}$$

$$w = w' = -b_1c_2 \tag{79}$$

Efficiency Circles.—In the charts where loss circles may be drawn, loci of constant efficiency of transmission may also be located. These loci will also be circles.

The efficiency is the ratio of power received to power supplied. The per cent efficiency η is consequently expressed by

$$\eta = \frac{P_r}{P_s}100 = \frac{P_r}{P_r + L_r}100 \tag{80}$$

Substituting for L_r by equation (55), equation (80) may be written

$$\frac{100}{\eta}P_r = P_r + L_r = P_r + uE_r^2 + tP_r - wQ_r + v\frac{P_r^2 + Q_r^2}{E_r^2} \tag{81}$$

Which, upon rearrangement, becomes

$$P_r^2 + \frac{t - \left(\frac{100}{\eta} - 1\right)}{v}E_r^2P_r + Q_r^2 - \frac{w}{v}E_r^2Q_r = -\frac{u}{v}E_r^4 \tag{82}$$

Adding

$$\left[\frac{t - \left(\frac{100}{\eta} - 1\right)}{2v}\right]^2 E_r^4 + \left[\frac{w}{2v}\right]^2 E_r^4 = \frac{\left[t - \left(\frac{100}{\eta - 1}\right)\right]^2 + w^2}{4v^2}E_r^4 \tag{83}$$

in order to complete the squares gives

$$\left[P_r + \frac{t - \left(\frac{100}{\eta} - 1\right)}{2v}E_r^2\right]^2 + \left[Q_r - \frac{w}{2v}E_r^2\right]^2 =$$
$$\left[\frac{E_r^2}{2v}\right]^2 \left\{\left[t - \left(\frac{100}{\eta} - 1\right)\right]^2 + w^2 - 4uv\right\} \tag{84}$$

For a three-phase system, equation (84) may be modified as follows:

$$\left[P_R + \frac{t - \left(\frac{100}{\eta} - 1\right)}{2{,}000v}E_R^2\right]^2 + \left[Q_R - \frac{w}{2{,}000v}E_R^2\right]^2 =$$
$$\left[\frac{E_R^2}{2{,}000v}\right]^2 \left\{\left[t - \left(\frac{100}{\eta} - 1\right)\right]^2 + w^2 - 4uv\right\} \tag{85}$$

As seen, this is the equation of a circle whose radius depends upon the value of the efficiency. The horizontal displacement of the center is also a function of the efficiency while the vertical dis-

placement is fixed (assuming constant receiver voltage). The coordinates of the center and the radius are given by

$$A_{R\eta} = -\frac{t - \left(\frac{100}{\eta} - 1\right)}{2{,}000v} E_R^2 \tag{86}$$

$$B_{R\eta} = \frac{w}{2{,}000v} E_R^2 \tag{87}$$

$$C_{R\eta} = \frac{E_R^2}{2{,}000v} \sqrt{\left[t - \left(\frac{100}{\eta} - 1\right)\right]^2 + w^2 - 4uv} \tag{88}$$

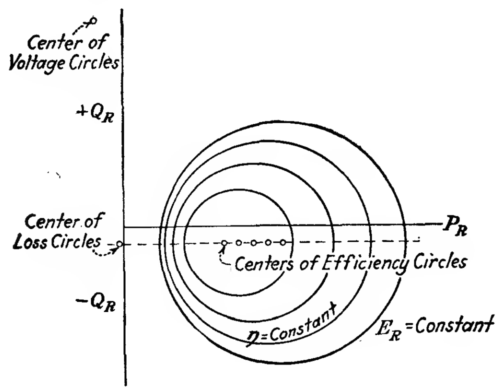

FIG. 161.—This figure shows how loci of constant efficiency of transmission can be located in the receiver chart for constant receiver voltage. These loci are circles which are eccentric but whose centers all lie on a straight line parallel to the active-power axis. From this diagram the efficiency of transmission can be determined when the active and reactive power received are known.

Ordinarily, $A_{R\eta}$ will be positive while $B_{R\eta}$ will be negative. Hence, the center (or centers) will lie in the fourth quadrant. It may be noted that the vertical displacement is the same as for the center of the loss circles. By assuming values of the efficiency a family of efficiency circles may be drawn as shown in Fig. 161. The circles are eccentric. Their centers, however, all lie on a straight horizontal line which also passes through the center of the loss circles.

It is seen that the greatest efficiency for any value of power received is realized when the reactive kilovolt-amperes at the receiver end are lagging and equal to $B_{R\eta}$, the vertical displacement of the efficiency centers. Evidently, *maximum efficiency* is obtained at the point where the *radius of the efficiency circles becomes zero.* This means that

$$\left[t - \left(\frac{100}{\eta_{\max}} - 1\right)\right]^2 + w^2 - 4uv = 0 \tag{89}$$

from which

$$\eta_{\max} = \frac{100}{t + 1 \pm \sqrt{4uv - w^2}} \tag{90}$$

The maximum efficiency may then be computed from equation (90) using the *plus sign in front of the radical.* Inserting the value of maximum efficiency in equation (86) gives the power at which it occurs. The reactive power at maximum efficiency is obviously given by equation (87).

Efficiency circles may also be drawn in the sending chart for constant sending voltage. The equation for these circles may be derived by expressing the efficiency as follows:

$$\eta = \frac{P_s - L_s}{P_s}100 \tag{91}$$

Omitting intermediate steps, this equation may readily be brought into the following form:

$$\left[P_s - \frac{t + \left(1 - \frac{\eta}{100}\right)}{2v'}E_s^2\right]^2 + \left[Q_s + \frac{w'}{2v'}E_s^2\right]^2 = \left[\frac{E_s^2}{2v'}\right]^2\left\{\left[t + \left(1 - \frac{\eta}{100}\right)\right]^2 + (w')^2 - 4u'v'\right\} \tag{92}$$

which, for a three-phase system, may be written

$$\left[P_S - \frac{t + \left(1 - \frac{\eta}{100}\right)}{2{,}000v'}E_S^2\right]^2 + \left[Q_S + \frac{w'}{2{,}000v'}E_S^2\right]^2 = \left[\frac{E_S^2}{2{,}000v'}\right]^2\left\{\left[t + \left(1 - \frac{\eta}{100}\right)\right]^2 + (w')^2 - 4u'v'\right\} \tag{93}$$

The displacement of the center and the radius are given by

$$A_{S\eta} = \frac{t + \left(1 - \frac{\eta}{100}\right)}{2{,}000v'}E_S^2 \tag{94}$$

$$B_{S\eta} = -\frac{w'}{2{,}000v'}E_S^2 \tag{95}$$

$$C_{S\eta} = \frac{E_S^2}{2{,}000v'}\sqrt{\left[t + \left(1 - \frac{\eta}{100}\right)\right]^2 + (w')^2 - 4u'v'} \tag{96}$$

The center is usually located in the first quadrant. Figure 162 shows a family of efficiency circles located in the sending chart for constant sending voltage.

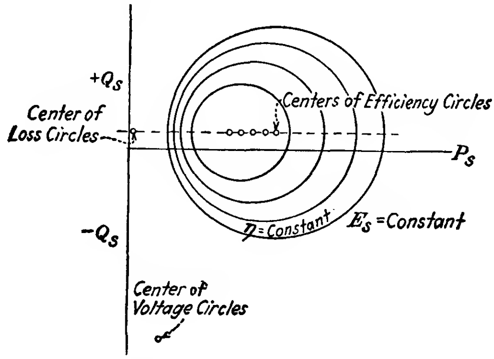

FIG. 162.—This figure shows how loci of constant efficiency of transmission can be located in the sending chart for constant sending voltage. The loci are circles which are eccentric but whose centers all lie on a straight line parallel to the active-power axis. From this diagram the efficiency of transmission can be determined when the active and reactive power at the sending end are known.

The maximum efficiency may also be calculated from equation (96) by putting

$$\left[t + \left(1 - \frac{\eta_{\max}}{100}\right)\right]^2 + (w')^2 - 4u'v' = 0 \tag{97}$$

the solution of which is

$$\eta_{\max} = 100[t + 1 \pm \sqrt{4u'v' - (w')^2}] \tag{98}$$

Here the minus sign should always be used before the radical. By inserting the value of the maximum efficiency in equation (94), the corresponding value of sending power may be determined.

Evidently, the efficiency calculated from equations (90) and (98) should be the same. That this actually will be the case may be shown as follows:

From equation (90),

$$\eta_{\max} = \frac{100}{t + 1 + \sqrt{4uv - w^2}} = \frac{100[t + 1 - \sqrt{4uv - w^2}]}{(t + 1)^2 - (4uv - w^2)} \tag{99}$$

By resorting to the original expressions for t, u, v, w, u', v', and w', it can be demonstrated that

$$4uv - w^2 = 4u'v' - (w')^2 \tag{100}$$

and that

$$(t + 1)^2 - (4uv - w^2)^2 = 1 \tag{101}$$

Hence, equation (99) reduces to equation (98) when the minus sign is used in the latter, thus giving the same maximum efficiency.

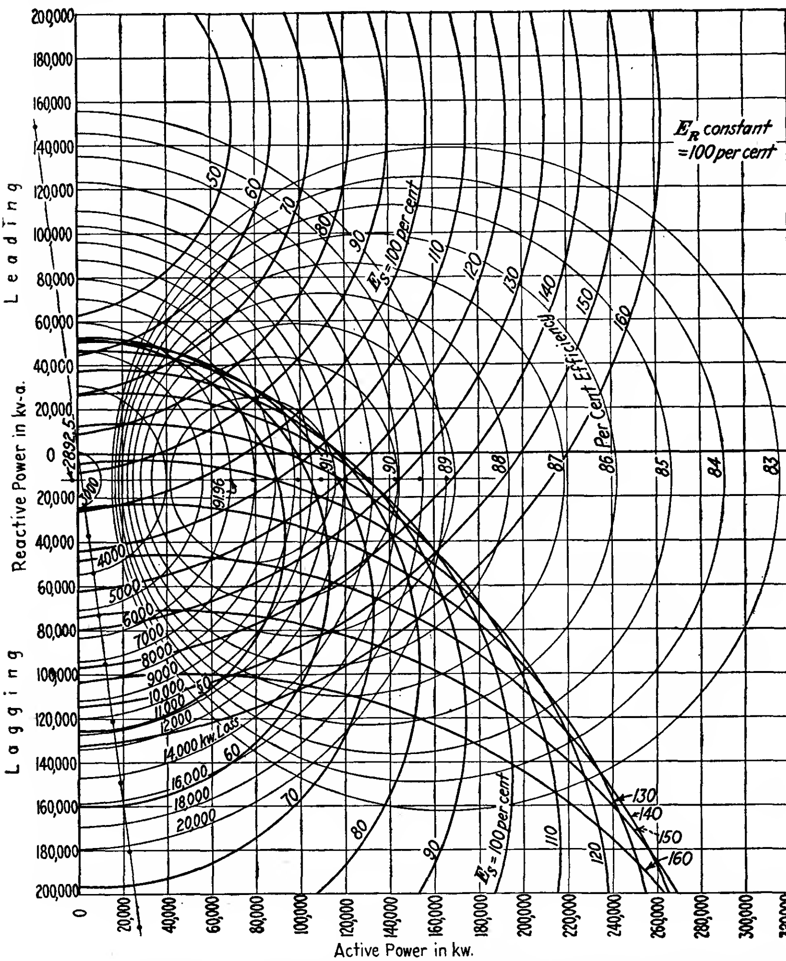

FIG. 163.—Evans and Sels sending and receiving chart for constant receiver voltage. This chart is for a single circuit of a 240-mile, 220-kv., three-phase, three-circuit transmission line with 360,000 kv.-a. of transformers at each end. The calculations for this chart will be found in Example 2, Parts (*a*) and (*b*).

Some Additional Remarks. Negative Power.—In constructing the Evans and Sels charts, it is possible and sometimes convenient to draw sending and receiver charts for either constant sending voltage or constant receiving voltage on the same sheet of paper.

In other words, charts of the type shown in Fig. 153 and in Fig. 156 and also the charts shown in Fig. 154 and Fig. 155 may be combined.

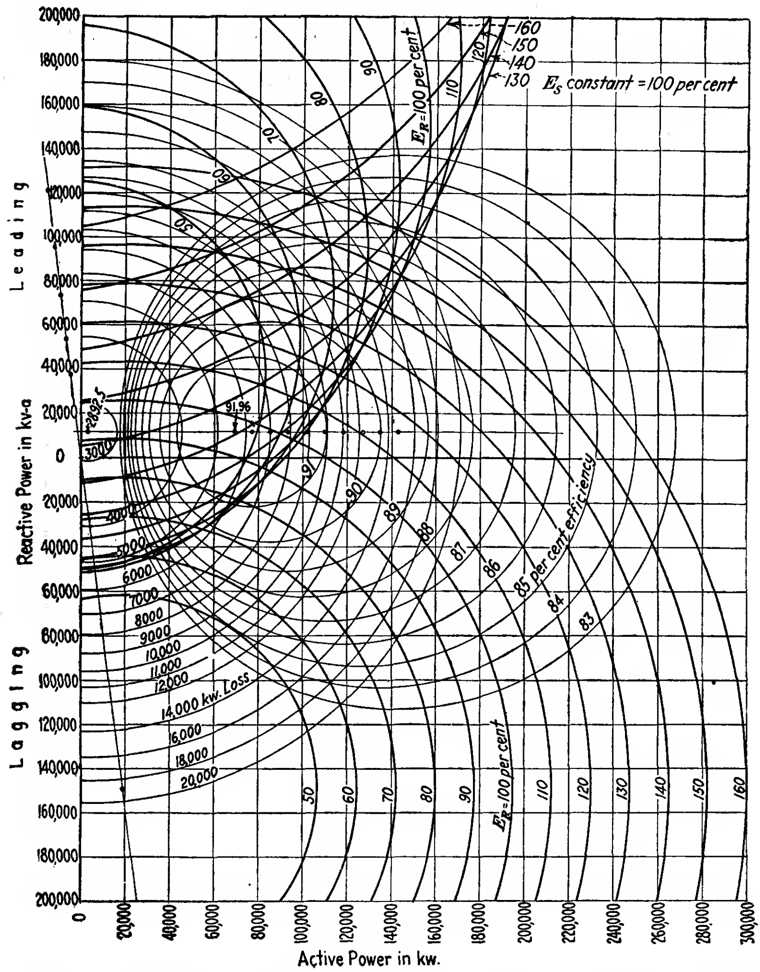

FIG. 164.—Evans and Sels sending and receiving chart for constant sending voltage. This chart is for a single circuit of a 240-mile, 220-kv., three-phase, three-circuit transmission line with 360,000 kv.-a. of transformers at each end. The calculations for this chart will be found in Example 2, Parts (*a*) and (*b*).

Figures 163 and 164 show two complete Evans and Sels charts in which the sending and receiving charts have been drawn on the same sheet. *It should be noted that the loss and efficiency must be read at the receiver operating point in the chart for constant receiver*

voltage, and at the sending operating point in the chart for constant sending voltage. It is unnecessary to have loss and efficiency circles for the other part of the chart. Furthermore they would complicate the chart in a prohibitive manner, since a complete family of these circles would be required for each value of the variable voltage. Complete computations for these charts will be found in Example 2.

In the charts so far, only positive values of power have been considered. The charts, however, may easily be extended to embrace also the negative-power region. This merely involves completing the voltage and loss circles and drawing a set of efficiency circles in the negative-power region. These efficiency circles are exact images of those for positive power. Negative values of power simply indicate that power is flowing in the opposite direction.

Whether or not the charts should be drawn to include also negative power depends on the particular problem whose solution is desired. Very often a mere inspection of the problem will reveal whether negative values of power are likely to be encountered. In other cases it may be quite difficult, if not impossible, to predict definitely the direction of power flow in all lines involved *under all conditions*. Obviously, in such cases it is safer to make the charts complete.

The Modified Evans and Sels Chart.—The Evans and Sels charts just described were all based on constant voltage at one end of the system. In order to study, therefore, the performance of the system at several values of sending- and receiving-end voltage, an operation which is frequently required, it would be necessary to construct a series of charts so as to cover the desired voltage range. This means that, in the first place, a considerable amount of work would have to be spent in the preparation of the charts, and, secondly, that the subsequent handling and use of the many charts would be rather inconvenient.

Fortunately, it is possible to avoid these difficulties by a modification of the original Evans and Sels chart.[1] The result is

[1] In discussions by F. E. Terman (*Trans. A.I.E.E.*, p. 84, 1924) and C. A. Nickle (*Trans. A.I.E.E.*, p. 85, 1924) attention was called to the possibility of "modifying" the usual circle diagrams so as to make each chart applicable at any value of sending and receiving voltage.

The modified charts are also briefly described in the paper "Power System Transients" by V. Bush and R. D. Booth, *Trans. A.I.E.E.*, p. 80, 1925.

a type of chart which is applicable to any value of voltage at the sending, as well as the receiving end of the system. Only two charts, one sending chart and one receiving chart, are required. Obviously this can be accomplished only by introducing a variable power scale depending on the voltage. The equations for the modified chart are easily derived from those previously established for the Evans and Sels chart, namely, equation (34) and (47).

Voltage Circles.—Dividing equation (34) by E_R^4 gives

$$\left[\frac{P_R}{E_R^2} + \frac{l}{1{,}000}\right]^2 + \left[\frac{Q_R}{E_R^2} + \frac{m}{1{,}000}\right]^2 = \left(\frac{n}{1{,}000}\right)^2\left(\frac{E_S}{E_R}\right)^2 \quad (102)$$

This is the equation of a circle in a coordinate system *where active and reactive power received, respectively, divided by the receiver voltage squared are the variables.* The position of the center is, as seen, *entirely independent of the voltages* and the radius is not given by absolute voltage values but depends on *the ratio of the sending and the receiving voltage.* A circle diagram based on equation (102) would be applicable at all voltages. The circles of voltage ratios would be concentric and evenly spaced, making interpolation easy. The scale would be correct and hence the chart direct-reading at *unity receiver voltage.* At other voltages the chart reading must be multiplied by the actual receiver voltage squared.

It is more convenient, however, to have the chart direct-reading at the nominal value (V_R) of the receiver voltage, rather than at 1 volt. This is readily accomplished by multiplying equation (102) by V_R^4. The result is

$$\left[\frac{P_R}{\left(\frac{E_R}{V_R}\right)^2} + \frac{l}{1{,}000}V_R^2\right]^2 + \left[\frac{Q_R}{\left(\frac{E_R}{V_R}\right)^2} + \frac{m}{1{,}000}V_R^2\right]^2 = \left[\frac{n}{1{,}000}V_R^2\right]^2\left(\frac{E_S}{E_R}\right)^2 \quad (103)$$

It is seen that the chart now will be direct-reading when $E_R = V_R$, *i.e.*, when the actual receiver voltage is equal to the nominal receiver voltage. For other values of receiver voltage, the scales should be multiplied by the square of the ratio of the actual receiver voltage to the nominal receiver voltage. The displacement of the center and the radius are given by

$$A_R = -\frac{l}{1{,}000}V_R^2 = -\frac{1}{1{,}000}\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}V_R^2 \qquad (104)$$

$$B_R = -\frac{m}{1{,}000}V_R^2 = -\frac{1}{1{,}000}\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}V_R^2 \qquad (105)$$

$$C_R = \frac{n}{1{,}000}V_R^2\frac{E_S}{E_R} = \frac{1}{1{,}000}\frac{V_R^2}{\sqrt{b_1^2 + b_2^2}}\frac{E_S}{E_R} \qquad (106)$$

The equation for the modified sending chart is obtained by multiplying equation (47) by V_S^4/E_S^4 where V_S is the nominal voltage at the sending end. The following equation is obtained:

$$\left[\frac{P_S}{\left(\frac{E_S}{V_S}\right)^2} - \frac{l'}{1{,}000}V_S^2\right]^2 + \left[\frac{Q_S}{\left(\frac{E_S}{V_S}\right)^2} - \frac{m'}{1{,}000}V_S^2\right]^2 = \left[\frac{n'}{1{,}000}V_S^2\right]^2\left(\frac{E_R}{E_S}\right)^2 \qquad (107)$$

Here the chart is direct-reading when the actual sending voltage is equal to the nominal sending voltage. The circles represent ratios of receiver voltage to sending voltage. The displacement of the center and the radius are given by

$$A_S = \frac{l'}{1{,}000}V_S^2 = \frac{1}{1{,}000}\frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2}V_S^2 \qquad (108)$$

$$B_S = \frac{m'}{1{,}000}V_S^2 = \frac{1}{1{,}000}\frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2}V_S^2 \qquad (109)$$

$$C_S = \frac{n'}{1{,}000}V_S^2\frac{E_R}{E_S} = \frac{1}{1{,}000}\frac{V_S^2}{\sqrt{b_1^2 + b_2^2}}\frac{E_R}{E_S} \qquad (110)$$

Figures 165 and 166 show two complete modified Evans and Sels charts. In addition to the voltage circles, radial angle lines have also been drawn in. Obviously, the modified charts are "angle true," just as well as the original Evans and Sels charts. In the modified charts, however, *only one family of angle lines is required since all voltage circles are concentric.* Hence, with these charts it is exceedingly convenient to make use of the angular relations. A point may be transferred from the sending chart to the receiving chart and *vice versa* by means of voltage ratio and angle, a process which is both direct and precise.

Although loss and efficiency circles easily can be drawn in, they are actually superfluous since, if desired, the loss can always be obtained as the difference between sending and receiving power, and the efficiency as the ratio of the receiving and sending power. It is always well to avoid unnecessary lines and to keep the charts

as simple as possible. As a matter of completeness, however, the necessary formulas for loss and efficiency circles are given below.

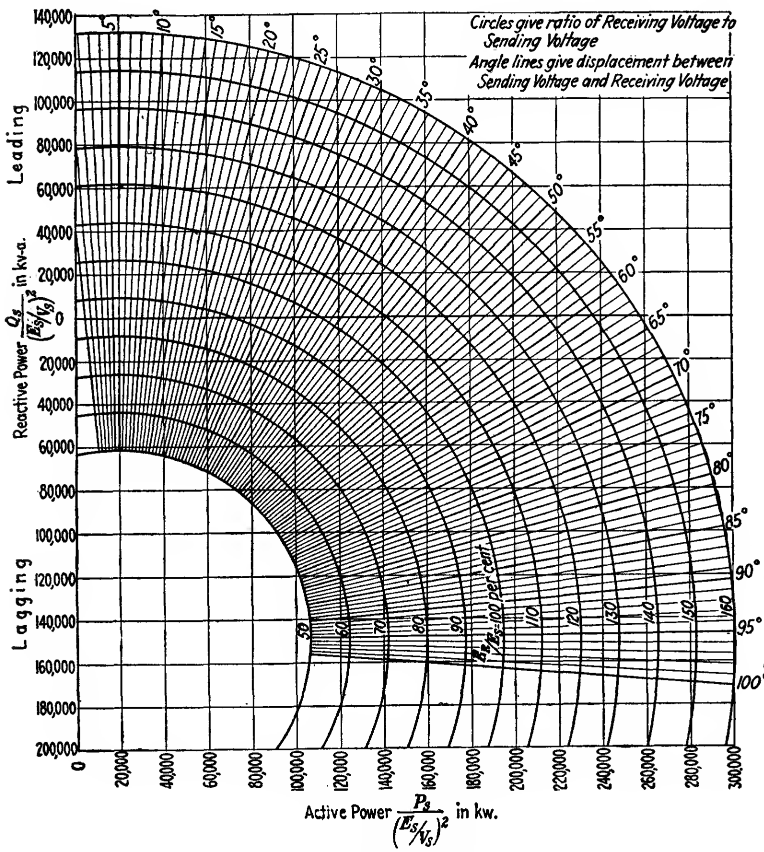

FIG. 165.—Modified Evans and Sels sending chart. This chart is for a single circuit of a 240-mile, 220-kv., three-phase, three-circuit transmission line with 360,000 kv.-a. of transformers at each end. The calculations for this chart will be found in Example 2, Parts (a) and (c).

Loss Circles.—By multiplying equation (59) by V_R^4/E_R^4 the equation for the loss circles in the modified Evans and Sels receiver chart is obtained as

$$\left[\frac{P_R}{(E_R/V_R)^2} + \frac{t}{2{,}000v}V_R^2\right]^2 + \left[\frac{Q_R}{(E_R/V_R)^2} - \frac{w}{2{,}000v}V_R^2\right]^2 = \left(\frac{V_R}{1{,}000}\right)^2\left[\frac{L_R}{v\left(\frac{E_R}{V_R}\right)^2} + \frac{t^2 + w^2 - 4uv}{4v^2}V_R^2\right] \quad (111)$$

This equation represents a circle whose center is fixed independent of the actual receiver voltage in a coordinate system with

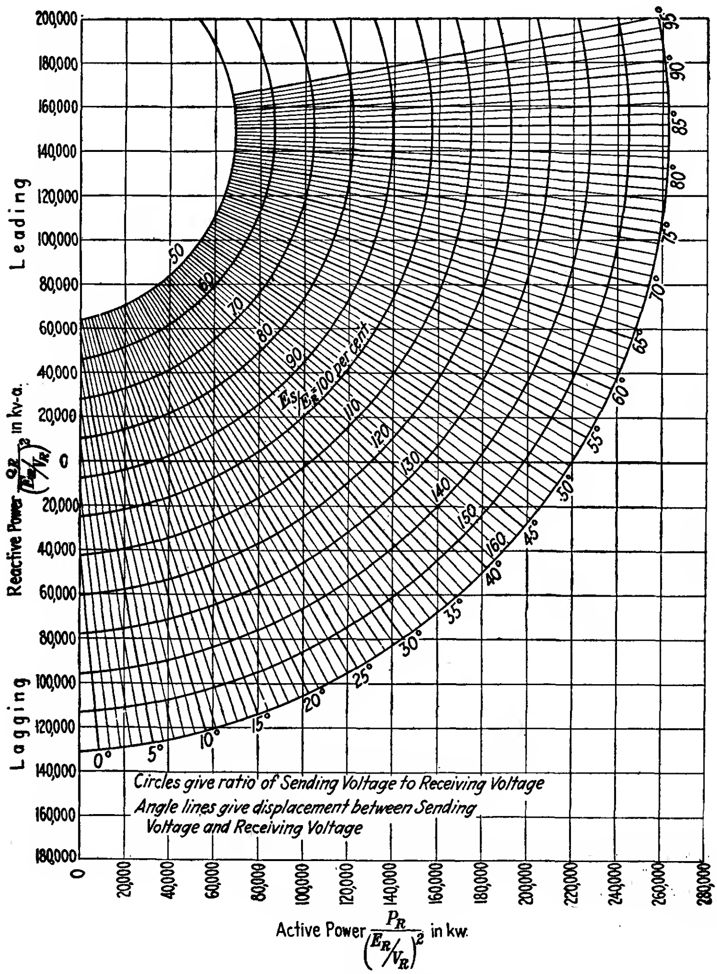

FIG. 166.—Modified Evans and Sels receiving chart. This chart is for a single circuit of a 240-mile, 220-kv., three-phase, three-circuit transmission line with 360,000 kv.-a. of transformers at each end. The calculations for this chart will be found in Example 2, Parts (*a*) and (*c*).

variables $\frac{P_R}{(E_R/V_R)^2}$ and $\frac{Q_R}{(E_R/V_R)^2}$. The values of the loss L_R indicated by the loss circles depend on the actual receiver voltage. The loss circles are direct-reading when the receiver voltage equals

the nominal voltage. In general, the values of loss as obtained from the circles should be multiplied by $(E_R/V_R)^2$.

The displacement of the center and the radius are given by

$$A_{RL} = -\frac{t}{2{,}000v}V_R^2 \tag{112}$$

$$B_{RL} = \frac{w}{2{,}000v}V_R^2 \tag{113}$$

$$C_{RL} = \frac{V_R}{1{,}000}\sqrt{\frac{L_R}{v\left(\frac{E_R}{V_R}\right)^2} + \frac{t^2 + w^2 - 4uv}{4v^2}V_R^2} \tag{114}$$

The equation for the loss circles in the modified sending chart is obtained by multiplying equation (69) by V_S^4/E_S^4 and becomes

$$\left[\frac{P_S}{(E_S/V_S)^2} - \frac{t}{2{,}000v'}V_S^2\right]^2 + \left[\frac{Q_S}{(E_S/V_S)^2} + \frac{w'}{2{,}000v'}V_S^2\right]^2 = \left(\frac{V_S}{1{,}000}\right)^2\left[\frac{L_S}{v'\left(\frac{E_S}{V_S}\right)^2} + \frac{t^2 + (w')^2 - 4u'v'}{4(v')^2}V_S^2\right] \tag{115}$$

The displacement of the loss-circle center in the sending chart nd the radius are given by

$$A_{SL} = \frac{t}{2{,}000v'}V_S^2 \tag{116}$$

$$B_{SL} = -\frac{w'}{2{,}000v'}V_S^2 \tag{117}$$

$$C_{SL} = \frac{V_S}{1{,}000}\sqrt{\frac{L_S}{v'\left(\frac{E_S}{V_S}\right)^2} + \frac{t^2 + (w')^2 - 4u'v'}{4(v')^2}V_S^2} \tag{118}$$

Efficiency Circles.—The equation for the efficiency circles in the modified Evans and Sels receiver chart is obtained by multiplying equation (85) by V_R^4/E_R^4. It becomes

$$\left[\frac{P_R}{(E_R/V_R)^2} + \frac{t - \left(\frac{100}{\eta} - 1\right)}{2{,}000v}V_R^2\right]^2 + \left[\frac{Q_R}{(E_R/V_R)^2} - \frac{w}{2{,}000v}V_R^2\right]^2 = \left[\frac{V_R^2}{2{,}000v}\right]^2\left\{\left[t - \left(\frac{100}{\eta} - 1\right)\right]^2 + w^2 - 4uv\right\} \tag{119}$$

As seen, this equation represents a circle. The position of the center and the radius of the circles are independent of the actual receiver voltage. The horizontal displacement of the center and

also the radius, however, depend on the value of the efficiency. The coordinates of the center and the radius are given by

$$A_{R\eta} = -\frac{t - \left(\frac{100}{\eta} - 1\right)}{2{,}000v} V_R^2 \tag{120}$$

$$B_{R\eta} = \frac{w}{2{,}000v} V_R^2 \tag{121}$$

$$C_{R\eta} = \frac{V_R^2}{2{,}000v}\sqrt{t - \left(\frac{100}{\eta} - 1\right)^2 + w^2 - 4uv} \tag{122}$$

By multiplying equation (93) by V_S^4/E_S^4, the equation for the efficiency circles in the modified sending chart becomes

$$\left[\frac{P_S}{(E_S/V_S)^2} - \frac{t + \left(1 - \frac{\eta}{100}\right)}{2{,}000v'} V_S^2\right]^2 + \left[\frac{Q_S}{(E_S/V_S)^2} + \frac{w'}{2{,}000v'} V_S^2\right]^2 = \left[\frac{V_S^2}{2{,}000v'}\right]^2\left\{\left[t + \left(1 - \frac{\eta}{100}\right)\right]^2 + (w')^2 - 4u'v'\right\} \tag{123}$$

This displacement of the center and the radius are, hence,

$$A_{S\eta} = \frac{t + \left(1 - \frac{\eta}{100}\right)}{2{,}000v'} V_S^2 \tag{124}$$

$$A_{S\eta} = -\frac{w'}{2{,}000v'} V_S^2 \tag{125}$$

$$C_{S\eta} = \frac{V_S^2}{2{,}000v'}\sqrt{\left[t + \left(1 - \frac{\eta}{100}\right)\right]^2 + (w')^2 - 4u'v'} \tag{126}$$

EXAMPLE 2

This example illustrates the calculation and construction of the Evans and Sels and the modified Evans and Sels charts for a typical high-voltage transmission circuit.

Statement of Problem

A large hydroelectric generating station supplies power over a three-phase, 220-kv., three-circuit transmission line with step-up and step-down transformers. The frequency is 60 cycles per second. The nominal (or 100 per cent) voltage at each end is 220 kv. referred to the high-tension side.

Line data:

Three circuits, each completely transposed.
Conductors: 795,000 cir. mils, A.C.S.R.
Length = 240 miles.
Equivalent equilateral spacing = 29 ft.

$$R = 0.117 \text{ ohm per mile}$$
$$X = 0.831 \text{ ohm per mile}$$
$$G = 0.1 \times 10^{-6} \text{ mho per mile}$$
$$C = 0.01382 \times 10^{-6} \text{ farad per mile}$$

Transformer data:

Identical sending and receiving transformers.
Total installation at each end = 360,000 kv.-a.
Equivalent R = 0.5 per cent
Equivalent X = 12 per cent
Core loss = 0.6 per cent
Copper loss = 0.5 per cent
Exciting current = 5 per cent

a. Compute the general circuit constants for a single circuit of the line with terminal transformers.

b. Prepare sending and receiving charts of the Evans and Sels type for constant (100 per cent) sending voltage and constant (100 per cent) receiving voltage.

c. Prepare a sending and a receiving chart of the modified Evans and Sels type.

NOTE.—Each chart should be drawn for a single circuit of the line with terminal transformers.

Solution

Part (a).—Constants of a Single Line Circuit without Transformers:

$$z = 0.117 + j0.831 = 0.839\underline{/82^\circ.0} \text{ ohm per mile}$$
$$y = (0.1 + j377 \times 0.01382)10^{-6} = 5.21 \times 10^{-6}\underline{/88^\circ.9} \text{ mho per mile}$$
$$\alpha = \sqrt{zy} = \sqrt{0.839 \times 5.21 \times 10^{-6}\underline{/85^\circ.45}} = 0.00209\underline{/85^\circ.45} \text{ hyp per mile}$$
$$\theta = \alpha l = 240 \times 0.00209\underline{/85^\circ.45} = 0.5015\underline{/85^\circ.45} \text{ hyp}$$
$$Z_0 = \sqrt{\frac{z}{y}} = 10^3\sqrt{\frac{0.839}{5.21}\overline{\backslash 3^\circ.45}} = 401\overline{\backslash 3^\circ.45} \text{ ohms}$$
$$A = D = \cosh\theta = 0.8782\underline{/1^\circ.25} = 0.8781 + j0.01913$$
$$B = Z_0 \sinh\theta = 401\overline{\backslash 3^\circ.45} \times 0.4814\underline{/85^\circ.84} = 193.1\underline{/82^\circ.39}$$
$$= 25.64 + j191.5$$
$$C = \frac{\sinh\theta}{Z_0} = \frac{0.4814\underline{/85^\circ.84}}{401\overline{\backslash 3^\circ.45}} = 0.0012\underline{/89^\circ.29}$$
$$= (0.1491 + j11.99)10^{-4}$$

Transformer Constants:

Capacity per circuit = 120,000 kv.-a.

$$I = \frac{120{,}000}{\sqrt{3} \times 220} = 315 \text{ amp.}$$
$$R = \frac{0.005 \times 220{,}000}{\sqrt{3} \times 315} = 2.017 \text{ ohms}$$
$$X = \frac{0.12 \times 220{,}000}{\sqrt{3} \times 315} = 48.40 \text{ ohms}$$

$$G = \frac{0.006 \times 120{,}000 \times 1000}{220{,}000^2} = 1.488 \times 10^{-5} \text{ mho}$$

$$Y = \frac{0.05 \times 120{,}000 \times 1000}{220{,}000^2} = 12.40 \times 10^{-5} \text{ mho}$$

$$B = \sqrt{12.40^2 - 1.488^2} \times 10^{-5} = 12.31 \times 10^{-5} \text{ mho}$$

$$Z = R + jX = 2.017 + j48.40 \text{ ohms}$$

$$Y = G - jB = (1.488 - j12.31)10^{-5} \text{ mho}$$

$$ZY = (2.017 + j48.40)(1.488 - j12.31)10^{-5} = (598.6 + j47.18)10^{-5}$$

$$1 + ZY = 1.00599 + j0.000472$$

$$1 + 2ZY = 1.01197 + j0.000944$$

General Circuit Constants.—These include a single circuit of the line and the transformers at each end. The approximate formulas (see Chap. IX, equations (83)) are used in this computation.

$$\begin{aligned}
A_0 &= a_1 + ja_2 = A(1 + 2ZY) + BY + CZ(1 + ZY) \\
&= (0.8781 + j0.01913)(1.01197 + j0.000944) + (25.64 + j191.5) \\
&\quad (1.488 - 12.31)10^{-5} + (0.1491 + j11.99)10^{-4}(2.017 + j48.40) \\
&\quad (1.00599 + j0.000472) \\
&= 0.8542 + j0.02301 \\
B_0 &= b_1 + jb_2 = B + A2Z + CZ^2 \\
&= 25.64 + j191.5 + 2(0.8781 + j0.1913)(2.017 + j48.40) + \\
&\quad (0.1491 + j11.99)10^{-4}(2.017 + j48.40)^2 \\
&= 27.06 + j273.7 \\
C_0 &= c_1 + jc_2 = C(1 + ZY)^2 + A2Y(1 + ZY) + BY^2 \\
&= (0.1491 + j11.99)10^{-4}(1.00599 + j0.000472)^2 + \\
&\quad 2(0.8781 + j0.01913)(1.488 - j12.31)10^{-5}(1.00599 + j0.000472) + \\
&\quad (25.64 + j191.5)(1.488 - j12.31)^2 10^{-10} \\
&= (0.04539 + j0.9940)10^{-3} \\
D_0 &= d_1 + jd_2 = A(1 + 2ZY) + BY + CZ(1 + ZY) = A_o \\
&= 0.8542 + j0.02301
\end{aligned}$$

Check on constants

$$A_0D_0 - B_0C_0 = 1 + j0$$

Part (*b*).—Constants for Voltage Circles:

$$b_1^2 + b_2^2 = 27.06^2 + 273.7^2 = 75{,}662$$

$$\sqrt{b_1^2 + b_2^2} = \sqrt{75{,}662} = 275.07$$

$$l = l' = \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} = \frac{0.8542 \times 27.06 + 0.02301 \times 273.7}{75{,}662}$$

$$= \frac{29.412}{75{,}662} = 0.3887 \times 10^{-3}$$

$$m = m' = \frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2} = \frac{0.02301 \times 27.06 - 0.8542 \times 273.7}{75{,}662}$$

$$= \frac{-233.20}{75{,}662} = -3.082 \times 10^{-3}$$

$$n = n' = \frac{1}{\sqrt{b_1^2 + b_2^2}} = \frac{1}{275.07} = 3.635 \times 10^{-3}$$

The positions of the centers and the radii for both the sending and receiving charts are given in Table XII.

TABLE XII.—CALCULATION OF VOLTAGE-POWER CIRCLES IN EVANS AND SELS CHART (EXAMPLE 2). SENDING AND RECEIVING CHART. RECEIVER VOLTAGE CONSTANT = 220 KV.

Circle	E_S in per cent of 220 kv.	E_R in per cent of 220 kv.	E_S^2, (kilovolts)^2	E_R^2, (kilovolts)^2	$A_S = \frac{l'E_S^2}{1,000}$, kilowatts	$B_S = \frac{m'E_S^2}{1,000}$, kilovolt-amperes	$A_R = -\frac{lE_R^2}{1,000}$, kilowatts	$B_R = -\frac{mE_R^2}{1,000}$, kilovolt-amperes	$C_S = C_R = \frac{nE_SE_R}{1,000}$, kilovolt-amperes
1	160	100	123,904	48,400	48,170	−381,890	−18,820	149,180	281,530
2	150	100	108,900	48,400	42,330	−335,650	−18,820	149,180	263,940
3	140	100	94,864	48,400	36,880	−292,390	−18,820	149,180	246,340
4	130	100	81,796	48,400	31,790	−252,110	−18,820	149,180	228,740
5	120	100	69,696	48,400	27,090	−214,810	−18,820	149,180	211,150
6	110	100	58,564	48,400	22,770	−180,500	−18,820	149,180	193,550
7	100	100	48,400	48,400	18,820	−149,180	−18,820	149,180	175,960
8	90	100	39,204	48,400	15,240	−120,830	−18,820	149,180	158,360
9	80	100	30,976	48,400	12,040	− 95,470	−18,820	149,180	140,770
10	70	100	23,716	48,400	9,220	− 73,100	−18,820	149,180	123,170
11	60	100	17,424	48,400	6,770	− 53,700	−18,820	149,180	105,570
12	50	100	12,100	48,400	4,700	− 37,290	−18,820	149,180	87,980

When the sending voltage is constant = 220 kv., Table XII may still be used, since the system is symmetrical ($A_0 = D_0$). The tabulated sending-end figures should be used for the receiver end and *vice versa*. Evidently, the figures indicating center positions will change sign.

Constants for Loss and Efficiency Circles:

$$\begin{aligned}
t &= 2(b_1c_1 + a_2d_2)\\
&= 2(27.06 \times 0.04539 \times 10^{-3} + 0.02301 \times 0.02301)\\
&= 3.516 \times 10^{-3}\\
u &= u' = a_1c_1 + a_2c_2\\
&= 0.8542 \times 0.04539 \times 10^{-3} + 0.02301 \times 0.9940 \times 10^{-3}\\
&= 61.65 \times 10^{-6}\\
v &= v' = d_1b_1 + d_2b_2\\
&= 0.8542 \times 27.06 + 0.02301 \times 273.7 = 29.412\\
w &= w' = b_2c_1 - b_1c_2 + a_1d_2 - a_2d_1\\
&= 273.7 \times 0.04539 \times 10^{-3} - 27.06 \times 0.9940 \times 10^{-3} = -14.471 \times 10^{-3}\\
w^2 - 4uv &= (w')^2 - 4u'v'\\
&= 14.471^2 \times 10^{-6} - 4 \times 61.65 \times 29.412 \times 10^{-6}\\
&= -7.043 \times 10^{-3}
\end{aligned}$$

$$\frac{t^2 + w^2 - 4uv}{4v^2} = \frac{t^2 + (w')^2 - 4u'v'}{4(v')^2}$$

$$= \frac{3.516^2 \times 10^{-6} - 7.043 \times 10^{-3}}{4 \times 29.412^2} = -2.032 \times 10^{-6}$$

The position of the centers and the radii of the loss and efficiency circles in the receiver chart for constant receiver voltage are given in Tables XIII and XIV. Since the system is symmetrical, Table XIII may be used also for the loss circles in the sending chart when the sending voltage is constant

= 220 kv., provided the signs of the horizontal and vertical displacement of the center are reversed. Table XV gives the necessary figures for efficiency circles in the sending chart.

TABLE XIII.—CALCULATION OF LOSS CIRCLES IN EVANS AND SELS CHART (EXAMPLE 2). RECEIVER CHART. RECEIVER VOLTAGE CONSTANT = 220 KV.

Circle	L_R, kilowatts	$A_{RL} = -\frac{tE_R^2}{2{,}000v}$, kilowatts	$B_{RL} = \frac{wE_R^2}{2{,}000v}$, kilovolt-amperes	C_{RL},[1] kilovolt-amperes
1	2,892.5	−2,893	−11,906	0
2	3,000	−2,893	−11,906	13,300
3	4,000	−2,893	−11,906	42,690
4	5,000	−2,893	−11,906	58,890
5	6,000	−2,893	−11,906	71,510
6	7,000	−2,893	−11,906	82,210
7	8,000	−2,893	−11,906	91,680
8	9,000	−2,893	−11,906	100,250
9	10,000	−2,893	−11,906	108,150
10	11,000	−2,893	−11,906	115,500

[1] $C_{RL} = \frac{E_R}{1{,}000}\sqrt{\frac{L_R}{v} + \frac{t^2 + w^2 - 4uv}{4v^2}E_R^2}.$

TABLE XIV.—CALCULATION OF EFFICIENCY CIRCLES IN EVANS AND SELS CHART (EXAMPLE 2). RECEIVER CHART. RECEIVER VOLTAGE CONSTANT = 220 KV.

Circle	η, per cent	$A_{R\eta} = -\frac{\left[t - \left(\frac{100}{\eta} - 1\right)\right]E_R^2}{2{,}000v}$ kilowatts	$B_{R\eta} = \frac{wE_R^2}{2{,}000v}$, kilovolt-amperes	$C_{R\eta}$,[1] kilovolt-amperes
1	91.96	69,050	−11,906	0
2	91.0	78,480	−11,906	37,300
3	90.0	88,530	−11,906	55,400
4	89.0	98,800	−11,906	70,660
5	88.0	109,310	−11,906	84,730
6	87.0	120,050	−11,906	98,210
7	86.0	131,050	−11,906	111,380
8	85.0	142,300	−11,906	124,430
9	84.0	153,830	−11,906	137,460
10	83.0	165,630	−11,906	150,550

[1] $C_{R\eta} = \frac{E_R^2}{2{,}000v}\sqrt{\left[t - \left(\frac{100}{\eta} - 1\right)\right]^2 + w^2 - 4uv}.$

TABLE XV.—CALCULATION OF EFFICIENCY CIRCLES IN EVANS AND SELS CHART (EXAMPLE 2). SENDING CHART. CONSTANT SENDING VOLTAGE = 220 KV.

Circle	η, per cent	$A_{S\eta} = \dfrac{\left[t + \left(1 - \frac{\eta}{100}\right)\right]E_S^2}{2{,}000v'}$ kilowatts	$B_{S\eta} = -\dfrac{w'E_S^2}{2{,}000\,v'}$, kilovolt-amperes	$C_{S\eta}$,[1] kilovolt-amperes
1	91.96	69,050	11,906	0
2	91.0	76,940	11,906	33,940
3	90.0	85,170	11,906	49,850
4	89.0	93,400	11,906	62,890
5	88.0	101,630	11,906	74,560
6	87.0	109,850	11,906	85,440
7	86.0	118,080	11,906	95,790
8	85.0	126,310	11,906	105,760
9	84.0	134,540	11,906	115,470
10	83.0	142,280	11,906	124,960

[1] $C_{S\eta} = \dfrac{E_S^2}{2{,}000v'}\sqrt{\left[t + \left(1 - \frac{\eta}{100}\right)\right]^2 + (w')^2 - 4u'v'}$.

Charts.—The finished Evans and Sels chart for constant (100 per cent) sending voltage and constant (100 per cent) receiving voltage are shown in Figs. 163 and 164, respectively. They are constructed from the data calculated in Tables XII to XV inclusive. As seen, sending charts and receiving charts have been drawn on the same sheet.

Part (*c*). Constants for Voltage Circles.—The values of the constants l, m, n, l', m', and n' will be the same as calculated in Part (*b*). Table XVI

TABLE XVI.—CALCULATION OF VOLTAGE-POWER CIRCLES IN MODIFIED EVANS AND SELS CHART (EXAMPLE 2). RECEIVER CHART

Circle	$\dfrac{E_S}{E_R}$, in per cent	V_R, kilovolts	V_R^2, (kilovolts)^2	$A_R = -\dfrac{lV_R^2}{1{,}000}$, kilowatts	$B_R = -\dfrac{mV_R^2}{1{,}000}$, kilovolt-amperes	$C_R = \dfrac{n}{1{,}000}V_R^2\dfrac{E_S}{E_R}$, kilovolt-amperes
1	160	220	48,400	−18,820	149,180	281,530
2	150	220	48,400	−18,820	149,180	263,940
3	140	220	48,400	−18,820	149,180	246,340
4	130	220	48,400	−18,820	149,180	228,740
5	120	220	48,400	−18,820	149,180	211,150
6	110	220	48,400	−18,820	149,180	193,550
7	100	220	48,400	−18,820	149,180	175,960
8	90	220	48,400	−18,820	149,180	158,360
9	80	220	48,400	−18,820	149,180	140,770
10	70	220	48,400	−18,820	149,180	123,170
11	60	220	48,400	−18,820	149,180	105,570
12	50	220	48,400	−18,820	149,180	87,980

gives the position of the center and the radii in the receiver chart for a nominal receiver voltage (V_R) of 220 kv.

Since the system is symmetrical and the nominal voltage at the sending end is the same as at the receiver end, the figures in Table XVI may be used also for the sending chart. The signs of the displacements of the center, however, should be reversed and the voltage ratio of the circles taken as E_R/E_S.

Charts.—The modified Evans and Sels charts, constructed from the data in Table XVI, are shown in Figs. 165 and 166. Only voltage circles and angular-displacement lines have been drawn in, while loss and efficiency circles are omitted. As stated in the discussion of the modified Evans and Sels chart, loss and efficiency circles are superfluous and only complicate the chart without being of any real advantage in its use.

CHAPTER XI

SYNCHRONOUS-MACHINE CHARTS

Very often it is convenient to determine the performance of synchronous machines by graphical methods. It is particularly important to have performance charts of the machines when it is desired to examine the operating characteristics of a system of which one or several generating stations form an integral part. Especially in problems involving stability analyses[1] it is necessary to obtain so many "operating points," *i.e.*, conditions which simultaneously satisfy all links in the system, such as lines, transformers, generators, synchronous condensers, and loads, that graphical solutions throughout are absolutely essential. If a study were to be made by analytical methods alone, the task would be very laborious and would require an undue amount of time, to say the least. When charts are available, however, the solution of even a problem of considerable complication may be obtained with comparative ease.

The synchronous machine charts[2] are essentially like the Evans and Sels charts for the transmission line discussed in the preceding chapter. The coordinates are power (kilowatts) and reactive power (reactive kilovolt-amperes). From a complete generator chart (for instance, Fig. 201) may be obtained the value of induced voltage (*air-gap voltage*) and field current for any assumed operating condition and also the angular displacement between the terminal and the induced voltage and between the terminal and the excitation voltage.

On account of the non-linearity of the magnetization curve resulting in a variable synchronous reactance, it is, in general, necessary to construct one chart for each value of terminal voltage which it is desired to consider. If it were possible to use a constant synchronous reactance, a single chart of the "modified type" would suffice for all terminal voltages. Unfortunately,

[1] Discussed in detail in Vol. II of this treatise.

[2] Briefly described in the paper, "Power System Transients," by V. Bush and R. D. Booth, *Trans. A.I.E.E.*, p. 80, 1925.

the assumption of a constant synchronous reactance will, as a rule, lead to inaccuracies larger than can be tolerated. Furthermore, it is also necessary to have separate charts for non-salient- and salient-pole machines, even though these machines may have the same magnetization curve, the same zero-power-factor characteristic, and the same short-circuit characteristic. This, of course, is due to the fact that, for power factors other than zero, the characteristic of the salient-pole machine differs materially from that of the non-salient-pole machine.

In constructing the charts, the resistance of the machines may or may not be taken into account. Usually the resistance of a synchronous generator is a small quantity and, if neglected, will not introduce appreciable error. Neglecting the resistance reduces the labor slightly in the computation of the data for the charts, and hence it is more or less a matter of judgment whether or not the resistance should be taken into account. A second advantage of neglecting the resistance is that a chart constructed for a generator may also be used to represent the same generator when operated as a synchronous motor, without completing the chart in the negative-power region.

Non-salient-pole Generators.—In any generator, a sinusoidal distribution of the magnetomotive force of the armature reaction

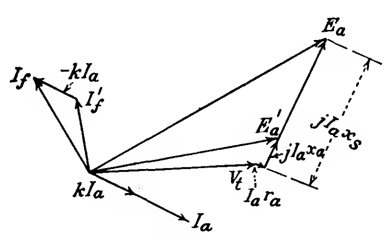

FIG. 167.—Vector diagram of a non-salient-pole generator supplying an inductive load.

may be assumed. In a non-salient-pole machine the reluctance of the air gap is constant, when the small effect of the slots is neglected. Hence, the distribution of the flux due to the armature reaction is also sinusoidal and may be considered as a vector.

When terminal voltage, power, and reactive power are given and the constants of the machine known, the calculation of induced voltage and excitation voltage is a simple problem which requires nothing but the solution of a simple vector diagram.

When computations have to be made, however, for a series of conditions, a lot of time can be saved by properly systematizing the work, putting the calculations into tabular form and using graphical methods during the process whenever possible. Figure 167 gives the well-known vector diagram of a non-salient-pole machine supplying an inductive load.

The following notation will be adopted:

V_t = terminal voltage
E'_a = voltage induced by air-gap flux
E_a = excitation voltage
I_a = armature current
P = active power at the terminals
Q = reactive power with respect to terminal voltage
P' = active power in the air gap
Q' = reactive power with respect to induced voltage (E'_a)
I_f = actual field current
I'_f = field current from magnetization curve corresponding to the induced voltage (E'_a)
$A = kI_a$ = armature reaction in terms of equivalent field current
x_a = armature leakage reactance
x_s = synchronous reactance

Sufficient data for the computations necessary for the charts may be obtained from the rating of the machine and the three characteristic curves, namely, the open-circuit, zero-power-factor, and short-circuit saturation curves. As a matter of fact, the two first-mentioned curves will usually suffice, and sufficient data may also be obtained from the first- and third-mentioned curves if the leakage reactance is given. During computations, it is usually most convenient to express all quantities on a percentage basis. Evidently, this is not necessary, but it permits the use of the same chart for generators of different ratings, provided they have the same fundamental characteristics.

Figure 168 shows the characteristic curves of a non-salient-pole generator. If necessary, the leakage reactance is determined by the Potier-triangle method. The relation between terminal voltage and the voltage induced by the air-gap flux is then given by

$$E'_a = V_t + jI_a x_a \tag{1}$$

The chart corresponding to this equation will be the ordinary Evans and Sels chart for a lumped reactance, the general circuit constants of which are

$$\left.\begin{aligned} A_0 &= a_1 + ja_2 = 1 \\ B_0 &= b_1 + jb_2 = jx_a \\ C_0 &= c_1 + jc_2 = 0 \\ D_0 &= d_1 + jd_2 = 1 \end{aligned}\right\} \quad (2)$$

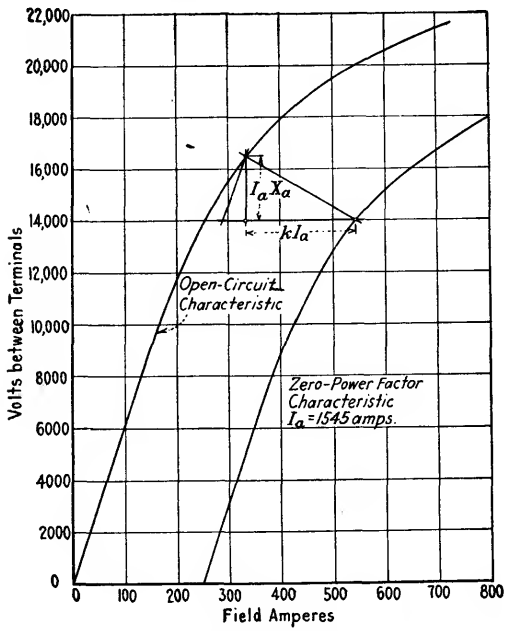

FIG. 168.—Characteristic curves of a 37,500-kv.-a., 14,000-volt, 60-cycle, 1,800-r.p.m. turbo generator.

Short-circuit ratio = 1.0
Armature leakage reactance = 18.5 per cent.

By referring to the general equations for the position of the center and radii of the sending-voltage circles in the Evans and Sels receiver chart (as given in Chap. X), the horizontal and vertical displacement of the center and the radii are, in this case, given by

$$A' = -\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} V_t^2 = 0 \quad (3)$$

$$B' = -\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2} V^2 = \frac{V_t^2}{x_a} \quad (4)$$

$$C' = \frac{E_a' V_t}{\sqrt{b_1^2 + b_2^2}} = \frac{E_a' V_t}{x_a} \quad (5)$$

The equation representing each of the constant E'_a circles obviously becomes

$$P^2 + \left(Q - \frac{V^2}{x_a}\right)^2 = \frac{(E'_a)^2 V_t^2}{x_a^2} \tag{6}$$

If the voltages are taken as volts to neutral and the leakage reactance is given in ohms per phase, the active power P and reactive power Q represent watts and volt-amperes per phase, respectively. If, on the other hand, per cent values of the voltages and the leakage reactance are used, the active and reactive power are given in per cent of the rating of the machine. As already stated, it is usually convenient to work on the percentage basis.

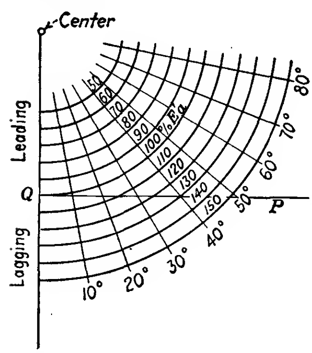

FIG. 169.—Generator chart showing circles of constant induced voltage and angle lines of constant displacement between terminal voltage and induced voltage. The resistance of the generator is neglected.

Figure 169 shows a chart based on equation (6). As seen, the center is displaced upward and lies on the vertical axis, since the resistance is neglected. It is drawn for constant terminal voltage and contains a family of circles, each representing a constant specified value of air-gap voltage E'_a. A family of radial angle lines is also drawn in so that the phase displacement between the terminal voltage and the induced voltage $\left(\angle\frac{E'_a}{V_t}\right)$ may be immediately read.

Usually, one chart has to be prepared for each value of terminal voltage, although, as far as the E'_a circles are concerned, a modified chart might be constructed so that the same chart could be used for all values of terminal voltage. Since, however, it is desired to put field-current curves on the same chart, this cannot, in general, be done, since the field currents are obtained from a non-linear magnetization curve.

The modified chart just referred to would be based on the following equation obtained by multiplying equation (6) by V_0^4/V_t^4

$$\left[\frac{P}{(V_t/V_0)^2}\right]^2 + \left[\frac{Q}{(V_t/V_0)^2} - \frac{V_0^2}{x_a}\right]^2 = \left(\frac{V_0^2}{x_a}\right)^2\left(\frac{E'_a}{V_t}\right)^2 \tag{7}$$

V_0 is the nominal terminal voltage of the machine and should, as a matter of convenience, be considered equal to the 100 per cent value. As seen, the circles in this chart represent ratios between the induced voltage and the terminal voltage. The power and reactive-power scales are not constant and depend on the actual value of the terminal voltage, the chart being direct-reading when the terminal voltage has its nominal value.

In order to obtain the lines of constant field current and angular displacement between the terminal voltage and the excitation voltage, it is necessary to calculate a sufficient number of points on each. These points are then laid off in the charts and smooth curves drawn in. The calculations are comparatively laborious but can be shortened by systematizing and by making use of an auxiliary chart, as described below.

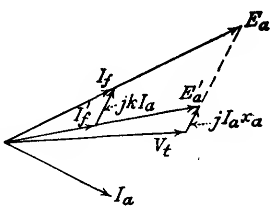

FIG. 170.—Vector diagram of a non-salient-pole generator with field-current and armature-reaction vectors rotated clockwise through 90 deg.

If the vectors in Fig. 167 representing field currents, and armature reaction are rotated clockwise through 90 deg., the diagram shown in Fig. 170 results. In this diagram, the field current I'_f, corresponding to the induced voltage, is in phase with the latter, and the vector kI_a, representing armature reaction, is in quadrature with the armature current. Also, the actual field current I_f is in phase with the excitation voltage E_a The relation between field current, field current corresponding to induced voltage, and armature reaction may, therefore, be expressed by

$$I_f = I'_f + jkI_a = I'_f + jk(I_{ap} + jI_{aq}) \tag{8}$$

This equation is easily reduced to

$$I_{ap}^2 + \left[I_{aq} - \frac{I'_f}{k}\right]^2 = \frac{I_f^2}{k^2} \tag{9}$$

In these equations, I_{ap} and I_{aq} represent components of the current in phase and in quadrature, respectively, with the induced voltage E'_a. Substituting the relations between internal power and internal reactive power as expressed by

$$P' = E'_a I_{ap} \quad \text{and} \quad Q' = E'_a I_{aq} \tag{10}$$

in equation (9), this reduces to

$$\left(\frac{P'}{E'_a}\right)^2 + \left(\frac{Q'}{E'_a} - \frac{I'_f}{k}\right)^2 = \frac{I_f^2}{k^2} \tag{11}$$

Dividing through by $(I'_f)^2$ gives

$$\left[\frac{P'}{E'_aI'_f}\right]^2 + \left[\frac{Q'}{E'_aI'_f} - \frac{1}{k}\right]^2 = \frac{1}{k^2}\left(\frac{I_f}{I'_f}\right)^2 \tag{12}$$

It is readily seen that this equation represents a circle and is of the "modified type." If $\frac{P'}{E'_aI'_f}$ and $\frac{Q'}{E'_aI'_f}$ are considered as the variable parameters, it is evident that a diagram based on this equation may be used for any value of I'_f, and instead of assigning specific values of the field current to the circles, they are lettered in terms of the ratio between the actual field current and the field current I'_f corresponding to induced voltage. The chart based on equation (12) will be an auxiliary chart, which is convenient during the process of gathering data for the actual chart. From the auxiliary chart, the actual field current corresponding to any assumed value of internal power and reactive power can be obtained. The values of the internal power P' and the internal reactive power Q' are given by

$$P' = P + \frac{P^2 + Q^2}{V_t^2}r_a \tag{13}$$

and

$$Q' = Q - \frac{P^2 + Q^2}{V_t^2}x_a \tag{14}$$

Evidently, if resistance is neglected, the internal and external power are identical. For any value of P and Q, E'_a is read from the circle diagram of induced voltages. The corresponding field current I'_f is obtained by entering the magnetization curve.

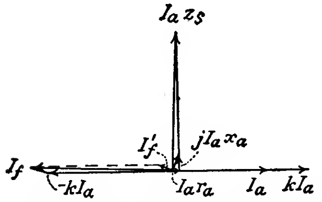

FIG. 171.—Vector diagram of a synchronous generator on short circuit.

The value of the constant k, which is representative of the armature reaction, may either be obtained from the open-circuit and the zero-power-factor characteristics or from the open- and short-circuit characteristics. Figure 171 gives a short-circuit vector diagram. From the magnetization curve I'_f, corresponding to the leakage-reactance drop I_ax_a, is obtained (see Fig. 168). The field current I_f corresponding to the same value of armature current is read either from the short-circuit curve or from the zero-power-factor curve at its point of intersection with the horizontal axis (at zero terminal

voltage). It is, in general, most convenient to use the rated (100 per cent) value of the armature current. The armature reaction in terms of equivalent field current is evidently $I_f - I'_f$. and the desired constant k becomes

$$k = \frac{I_f - I'_f}{I_a} \tag{15}$$

In order to introduce percentages in equation (12) the power (P') and reactive power (Q') should be taken in per cent of the rating. The air-gap voltage (E'_a) and the field currents (I_f and I'_f) should be taken as fractions; the former of the normal open-circuit voltage, the latter either of the rated armature current or of the field current corresponding to normal voltage on open circuit. If taken as a fraction of the rated armature current, k is calculated from equation (15) either by using the field currents and the armature current in amperes, or by expressing them both in terms of rated armature current. If the field currents are introduced as fractions of normal field current, k must be calculated from equation (15) with the field currents expressed in per cent of normal field current and the armature current in per cent of its rated value. The last-mentioned method is believed preferable and is used in the calculations in Example 1.

In the auxiliary chart, then, on a percentage basis, the horizontal and vertical displacement of the center and the radii of the circles are given by

$$A'' = 0 \tag{16}$$

$$B'' = \frac{100}{k} \tag{17}$$

$$C'' = \frac{100}{k}\left(\frac{I_f}{I'_f}\right) \tag{18}$$

Figure 172 shows the auxiliary chart. This chart also contains a family of radial lines which evidently represent angular displacement between I_f and I'_f or, what amounts to the same thing, between E_a and E'_a.

The process of computing data for the main generator chart, corresponding to a specific terminal voltage, may be summed up as follows: Determine x_a and k from the characteristic curves. Compute the location of the center for the E'_a circles, and draw in these circles. Also, draw in the family of radial lines indicating angular displacement between E'_a and V_t. In order to get points

on the field-current curves, proceed as follows: Choose values of P and Q; read the corresponding E'_a and $\angle\frac{E'_a}{V_t}$ from the main chart; and select the correct I'_f from the magnetization curve. Compute P' and Q' and read I_f and the angle between E_a and E'_a from the auxiliary chart. By selecting a number of values for P and Q, a sufficient number of points on the field-current curves are determined so that these curves may be properly located in the main chart. Similarly, a sufficient number of points on the angular-

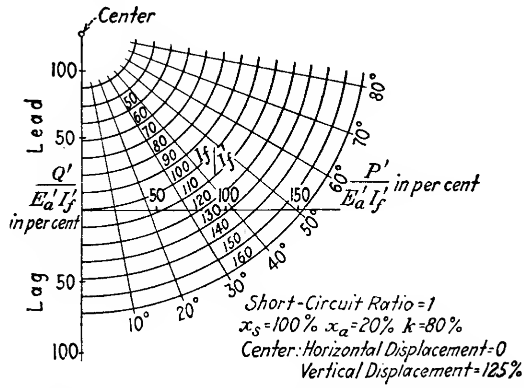

FIG. 172.—Auxiliary chart for a synchronous generator. When this chart is used for a non-salient-pole machine, the circles give directly the ratio of actual field current to field current corresponding to induced voltage. The angle lines give displacement between field current and field current corresponding to induced voltage or between excitation voltage and voltage induced by the air-gap flux.

displacement curves may be determined so that these curves may also be drawn in. In obtaining the necessary data, it is usually convenient to arrange the figures in tabular form, as suggested in Table XVII, in which, incidentally, some of the columns may be omitted if deemed superfluous.

The computations have to be repeated for each terminal voltage which it is desired to consider, and one main chart constructed for each terminal voltage. It is evident, however, that the same auxiliary chart may be used for all the computations, independent of terminal voltage.

The values of field current and angular displacement obtained in the tables will usually be non-integral. The curves in the main chart, however, should be for convenient integral values of field current as well as angular displacement. In order to obtain

TABLE XVII.—CALCULATION OF A GENERATOR CHART

V_t	P	Q	$\angle \frac{E'_a}{-V_t}$	E'_a	I'_f	P'	Q'	$\frac{P'}{E_a I'_f}$	$\frac{Q'}{E'_a I'_f}$	$\angle \frac{E_a}{-E'_a}$	$\frac{I_f}{I'_f}$	$\angle \frac{E_a}{-V_t}$	I_f

such integral values, the non-integral values may be plotted *versus* power for constant reactive kilovolt-amperes or *versus* reactive kilovolt-ampere for constant power. From these curves, the desired integral values may easily be read. Such auxiliary plots are indicated in Fig. 173.

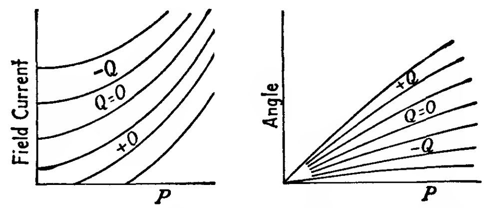

FIG. 173.—Auxiliary curves of field current and angular displacement between excitation voltage and terminal voltage plotted *versus* active power for constant reactive power. These curves make it possible to select convenient integral values of field current and angular displacement for location in the main generator charts.

Salient-pole Generators.—The reluctance which the armature reaction has to overcome in a salient-pole machine is not constant. Evidently, the reluctance of the paths in the interpolar spaces is considerably higher than the reluctance of the air gap under the pole shoes. The relative amounts of reaction flux, therefore, which will be set up in the two regions depend on the reluctances and on the power factor of the armature current with respect to the excitation voltage, since the latter fixes the space relation of the magnetomotive force of the armature reaction with respect to the field axis.

When this power factor is zero, the field axis and the axis of the armature magnetomotive force coincide, the result being a weakening of the main flux without distortion. When the power factor with respect to the excitation voltage is unity, the axis of the magnetomotive force of the armature lies midway between poles. Obviously, the result is then a distortion of the main flux without appreciable change in total strength.

In order to take the effect of the two reactions into account, it is convenient to resort to Blondel's two-reaction method.[1] For any angle ϕ between excitation voltage and armature current, the total armature reaction may be resolved into two components by multiplying the total reaction by $\cos\phi$ and $\sin\phi$, respectively. It is evident that the axis of the former will lie midway between poles, while the axis of the latter will coincide with the axis of the poles.

If there are Z inductors per pole per phase, all concentrated in a single pair of slots, the magnetomotive force in ampere-turns per pole per phase, due to the maximum phase current, is

$$\sqrt{2}\, I_a \frac{Z}{2} \tag{19}$$

where I_a is the effective current per phase in amperes. This magnetomotive force is uniform between the pair of slots considered, and its shape is rectangular, as shown in Fig. 174. The Fourier series, which represents the space distribution of this rectangular wave, is given by

$$\frac{4}{\pi}\left[\sqrt{2}\, I_a \frac{Z}{2}\right]\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right] \tag{20}$$

To simplify calculations, only the fundamental component will be considered. This will not introduce a very great error, especially in the case of three-phase machines, since, in this case, the sum of the third harmonics is zero.

[1] The derivation of the formulas for the demagnetizing and cross-magnetizing coefficients is taken from R. R. LAWRENCE, "Principles of Alternating-current Machinery," McGraw-Hill Book Company, Inc., New York, 1920.

Since this chapter was written, a very comprehensive discussion of synchronous-machine theory has been given in the paper, "Synchronous Machines," by R. E. DOHERTY and C. A. NICKLE. The paper is divided into four parts:

I. "An Extension of Blondel's Two-reaction Theory," *Trans. A.I.E.E.*, p. 912, 1926.

II. "Steady-state Power-angle Characteristics," *Trans. A.I.E.E.*, p. 927, 1926.

III. "Torque-angle Characteristics under Transient Conditions," *Jour. A.I.E.E.*, p. 1339, December, 1927.

IV. "Single-phase Short-circuits," *Jour. A.I.E.E.*, 1928.

As seen, Part I of the paper presents an extension of Blondel's two-reaction method. It is probable that the methods which Doherty and Nickle suggest will improve still further the precision of salient-pole-machine calculations.

The maximum ordinate of the fundamental is

$$\frac{4}{\pi}\sqrt{2}\,I_a\frac{Z}{2} = 0.9I_aZ \tag{21}$$

In the case where the winding is distributed instead of being concentrated, equation (21) should be multiplied by a reduction factor equal to the product of the pitch factor k_p and the breadth factor k_b.

Considering the armature reaction of any one phase as an oscillating vector and resolving it into two oppositely revolving vectors, each of one-half the magnitude of the oscillating vector, the reaction per phase may be looked upon as consisting of two rotating magnetomotive forces of sinusoidal space distribution. They both revolve with synchronous speed with respect to the armature. One of these component reactions will obviously be

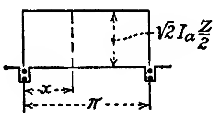

FIG. 174.

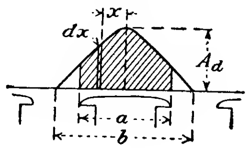

FIG. 175.

FIG. 174.—Rectangular distribution of magnetomotive force produced by a winding concentrated in a single pair of slots.

FIG. 175.—Diagram showing the demagnetizing component of the armature reaction. Its maximum coincides with the pole axis. The shaded portion of this magnetomotive force is effective in producing reaction.

stationary, while the other will rotate with twice synchronous speed with respect to the field poles. The maximum amplitude of each becomes

$$0.45k_pk_bI_aZ \tag{22}$$

In a three-phase machine, the reactions which revolve at double synchronous speed with respect to the field are displaced by 240 deg. and add up to zero. The reactions, however, which are stationary with respect to the field poles, coincide, and the effect of the three phases is obtained by adding the three reactions algebraically. Hence, the armature reaction of a three-phase machine is given by

$$0.45k_pk_bI_aZ3 \tag{23}$$

If Z is considered to be the total number of inductors per pole in all phases, this reduces to

$$0.45k_pk_bI_aZ \tag{24}$$

The direct or demagnetizing and the distorting or cross-magnetizing components are, respectively,

$$A_d = 0.45k_pk_bI_aZ \sin \phi \tag{25}$$

and

$$A_c = 0.45k_pk_bI_aZ \cos \phi \tag{26}$$

The curve representing the direct component of the reaction with respect to the pole is shown in Fig. 175. The magnetomotive force is not constant over the pole but varies according to the sine law. Since it is not constant, it is necessary to determine its average value over the pole shoe in order to find its effect in modifying the field. The portion of the magnetomotive force which is effective is shaded in the sketch. In reality, a small amount of fringing should be taken into account, but since this fringing is not very definite, it will be omitted. Designating the pole arc by a and the pole pitch by b, the mean value of the effective magnetomotive force is

$$A_d' = \frac{0.45k_pk_bI_aZ \sin \phi}{a\pi/b} \int_{-\frac{a\pi}{b2}}^{+\frac{a\pi}{b2}} \cos x dx$$

$$= 0.45k_pk_bI_aZ \sin \phi \frac{\sin \frac{a\pi}{b2}}{a\pi/b2} = K_d(k_pk_bI_aZ \sin \phi) \tag{27}$$

where

$$K_d = 0.45 \frac{\sin \frac{a\pi}{b2}}{a\pi/b2} \tag{28}$$

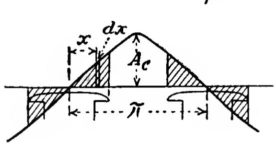

FIG. 176.

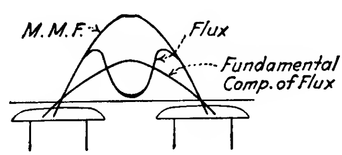

FIG. 177.

FIG. 176.—Diagram showing the cross-magnetizing component of the armature reaction. Its maximum is located midway between poles. The shaded portion of this magnetomotive force is effective in producing reaction.

FIG. 177.—Approximate flux distribution produced by the cross component of armature reaction. This flux curve contains a big third harmonic in addition to the fundamental component which is shown.

The coefficient K_d is called the *demagnetizing coefficient* and is equal to 0.45 times the factor by which the direct component of the reaction must be multiplied in order to take into account the ratio of pole arc to pole pitch.

The cross-magnetizing component, whose maximum value is given by equation (26), is shown in Fig. 176. The shaded portion of this figure shows the part of this magnetomotive force which is really effective. The actual shape of the flux produced by the cross-component is shown in Fig. 177. The cross-component tends to produce a local or component flux which passes across one-half of the air gap, then through the pole shoe, and back across the other half of the air gap. The circuit is then completed through the armature.

The effective part of the cross-magnetomotive force will be resolved into a Fourier series, and all terms above the fundamental neglected. The amplitude M of the fundamental is obtained by the following integration:

$$\begin{aligned} M &= \frac{2}{\pi}\int_0^{\pi} F(x)\sin x dx \\ &= \frac{2}{\pi}A_c\left[\int_0^{\frac{a\pi}{b2}}\sin^2 x dx + \int_{\pi-\frac{a\pi}{b2}}^{\pi}\sin^2 x dx\right] \\ &= A_c\left[\frac{a}{b} - \frac{1}{\pi}\sin\frac{a\pi}{b}\right] \end{aligned} \tag{29}$$

The mean value of this fundamental component is $2/\pi$ times the maximum value and, hence, equal to

$$\frac{2}{\pi}M = \frac{2}{\pi}A_c\left[\frac{a}{b} - \frac{1}{\pi}\sin\frac{a\pi}{b}\right] \tag{30}$$

The complete expression for the cross component of the armature reaction is

$$\begin{aligned} A_c' &= 0.45k_pk_bI_aZ\cos\phi\,\frac{2}{\pi}\left[\frac{a}{b} - \frac{1}{\pi}\sin\frac{a\pi}{b}\right] \\ &= K_c(k_pk_bZI_a\cos\phi) \end{aligned} \tag{31}$$

where

$$K_c = \frac{0.9}{\pi}\left[\frac{a}{b} - \frac{1}{\pi}\sin\frac{a\pi}{b}\right] \tag{32}$$

The factor K_c is called the *cross-magnetizing* coefficient. K_c is 0.45 times the factor by which the cross component of the armature magnetomotive force must be multiplied in order that it may be used on the open-circuit characteristic. The component flux due to the cross component of the reaction will probably not be as highly influenced by saturation as the flux due to the demagnetizing component. How much it will be affected by saturation

depends upon what parts of the magnetic circuit saturate first. If saturation primarily takes place in the stator teeth and in the pole faces, the two components may be about equally influenced by saturation. If, on the other hand, the yokes and the poles themselves saturate first, the component flux due to the cross-magnetizing reaction is only slightly affected by the degree of saturation. If in this last-mentioned case, therefore, it is desired to determine the voltage produced by it, the lower and less saturated part of the magnetization curve should be used. Finally, if all parts of the magnetic circuit saturate about equally, the cross flux is still affected by saturation although, obviously, to a somewhat smaller extent than the main flux.

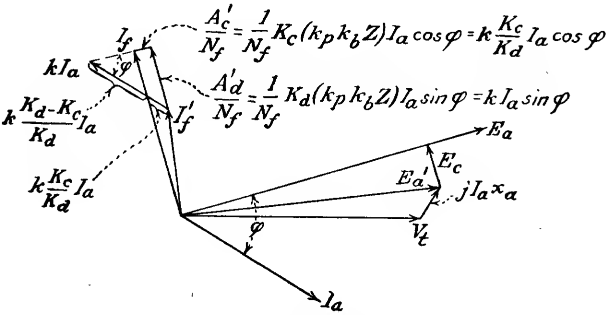

FIG. 178.—Vector diagram of a salient-pole generator supplying an inductive load.

Figure 178 shows the vector diagram for the salient-pole machine. The reactions are expressed in terms of equivalent field current. The total armature reaction is kI_a, and the vector representing it is drawn opposite in direction to the armature current from the end of the vector I_f', the field current corresponding to the induced voltage E_a'. The total field current is I_f; the demagnetizing component of armature reaction is drawn parallel, and the cross-magnetizing component of armature reaction, perpendicular to this field current. This method of treating the armature reaction assumes that the effect of the demagnetizing and the cross-magnetizing reaction *is equally affected by saturation.* Examination of the vector diagram will show that the field-current vector obviously divides the armature-reaction vector (kI_a) into two parts, the ratio of which is as $\frac{K_c}{K_d - K_c}$. Making use of this relation, the direction

of the total field-current vector is immediately given when the total armature-reaction vector is drawn. A perpendicular from the armature-reaction vector onto the field-current line immediately fixes the magnitude of the latter.

The value of the coefficient k representing the total armature reaction is obtained in the usual manner from an open-circuit and a zero-power-factor curve or else from an open-circuit and a short-circuit curve. In both cases, the power factor of the current with respect to the excitation voltage will be practically zero, and, hence, there will be no cross-magnetizing action. The coefficients K_d and K_c cannot very well be obtained from test data but are computed by the given formulas. The only information necessary for this is the ratio of pole arc to pole pitch.

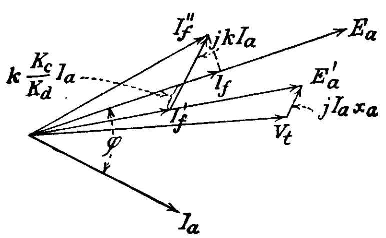

FIG. 179.—Vector diagram of a salient-pole generator with all field-current vectors rotated through 90 deg. in a clockwise direction.

Figure 179 shows the same vector diagram with all field-current vectors rotated through 90 deg. in a clockwise direction. It is evident that in obtaining data for the salient-pole charts, the same procedure as outlined for the non-salient-pole machines can, in general, be followed. The location of the center of the circles of constant air-gap voltage E_a' and their radii are determined by the same formulas (equations (3), (4), and (5)). Hence, this part of the chart will be the same for a non-salient- and a salient-pole machine having the same leakage reactance. The auxiliary chart can also be constructed in the manner described for the non-salient-pole machine, but when it comes to determining the magnitude and direction of the field current, a special device will have to be resorted to.

On a strip of paper or celluloid two scales may be constructed. The units are immaterial, but the ratio of the smaller to the larger scale should be as K_c:K_d. This strip may be pinned down at the origin as indicated in Fig. 180. A point corresponding to

internal power and reactive power is spotted in the auxiliary chart exactly as for a non-salient-pole machine. The strip of paper is now swung until its edge touches the located point, and the figure on the larger scale is read. Another strip of paper, which is pinned down at the center of the field-current circles, is then rotated until its edge coincides with the figure on the smaller scale corresponding to the figure just read on the larger scale. By means of a triangle, a perpendicular is dropped from the located point onto the edge of the strip rotating about the field-current center. At this point, the magnitude of the field current, as well as the angular displacement between the excitation voltage

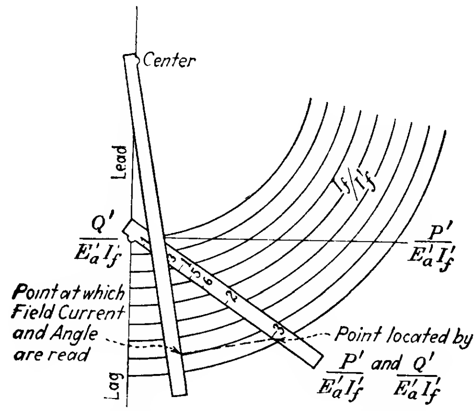

FIG. 180.—Auxiliary chart for a synchronous generator. This figure illustrates the method for obtaining values of field current and angular displacement in a salient-pole machine.

and the induced voltage, is read. This operation is evidently in accordance with the vector diagram and constitutes a very expedient method of determining field current of a salient-pole machine.

The data for the main charts are tabulated exactly as for the non-salient-pole generators, and the charts constructed in a similar manner. A set of charts for a salient-pole generator is shown in Figs. 199 to 204 inclusive. The actual computations for these will be found in Example 1.

Simplified Chart.—For non-salient-pole machines it is possible to construct a simplified chart[1] which is applicable at all terminal

[1] This chart as developed by J. M. BRYANT is described in Appendix I of a paper by C. A. NICKLE and F. L. LAWTON: "An Investigation of Transmission-system Power Limits," *Trans. A.I.E.E.*, p. 1, 1926.

voltages also as far as determination of field currents is concerned. This chart, however, is *approximate*.

The field current in a non-salient-pole machine may be obtained by vector addition of the field current as obtained from the magnetization curve corresponding to the terminal voltage and a field current which is proportional to the armature current. This latter will simultaneously take into account the effect of the armature reaction and the leakage-reactance drop, the resistance drop being neglected. Figure 181 shows a vector diagram utilizing this method of determining the field current.

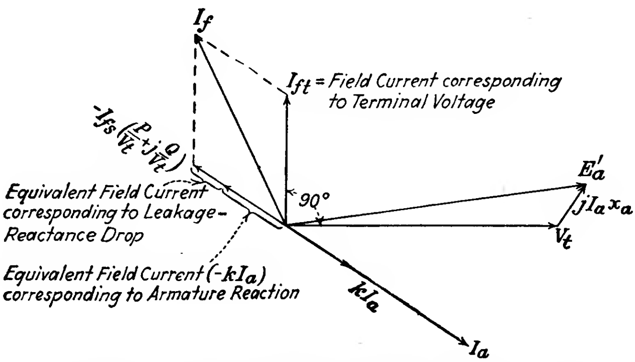

FIG. 181.—Approximate vector diagram of a non-salient-pole generator supplying an inductive load.

In setting up the equations for this chart, all quantities will be taken on a percentage basis. The following notation is used:

P = per cent active power
Q = per cent reactive power
V_t = per cent terminal voltage
I_{ft} = per cent field current to give the voltage V_t on open circuit as given by the magnetization curve
I_{fs} = per cent field current to give normal armature current on short circuit
I_f = per cent total field current
I_a = per cent armature current

Rotating the field-current vectors through 90 deg., as shown in Fig. 182, the field current under load is given by

$$I_f = I_{ft} + j\frac{I_{fs}I_a}{100} = I_{ft} + jI_{fs}\left(\frac{P}{V_t} + j\frac{Q}{V_t}\right) \qquad (33)$$

which may be modified to

$$\left(\frac{I_{fs}P}{V_t}\right)^2 + \left(I_{ft} - \frac{I_{fs}Q}{V_t}\right)^2 = I_f^2 \tag{34}$$

Dividing through by I_{ft}^2 gives

$$\left[\frac{P}{V_t I_{ft}/I_{fs}}\right]^2 + \left[\frac{Q}{V_t I_{ft}/I_{fs}} - 1\right]^2 = \left(\frac{I_f}{I_{ft}}\right)^2 \tag{35}$$

This equation represents a circle with variable parameters $\dfrac{P}{V_t I_{ft}/I_{fs}}$ and $\dfrac{Q}{V_t I_{ft}/I_{fs}}$, respectively. The circles represent constant ratios

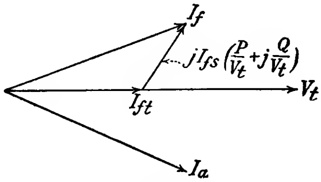

FIG. 182.—Approximate vector diagram of a non-salient-pole generator with field-current vectors rotated clockwise through 90 deg.

of actual field current and field current corresponding to the terminal voltage.

The horizontal and vertical displacement of the center and the radius are given by

$$A = 0 \tag{36}$$
$$B = 1 \tag{37}$$
$$C = \frac{I_f}{I_{ft}} \tag{38}$$

Figure 183 shows this chart. As seen, angle lines have also been drawn in giving displacements between excitation voltage and terminal voltage. Obviously, the chart has to be used in conjunction with the magnetization curve, since for any value of terminal voltage V_t the corresponding value of field current I_{ft} must be known.

As already stated, this chart should, in general, be looked upon as being approximate. If the machine is operating on the straight (or almost straight) part of the magnetization curve, it will give very good results. If, on the other hand, the machine operates at a high saturation where the magnetization curve is decidedly non-linear, inaccuracies may be anticipated, particularly

at low power factors. The reason for the discrepancies in this region is that the magneto-motive force (or equivalent field current) representing the effect of the leakage-reactance drop is always taken equal to its value at short circuit, *i.e.*, at a very low saturation. This value is too low for high saturations, as illustrated in Fig. 184. The chart utilizes the equivalent field current m at all saturations, while, at the higher saturation indicated, the equivalent field current of the leakage-reactance drop, at zero power factor (lagging), should appropriately have the value n.

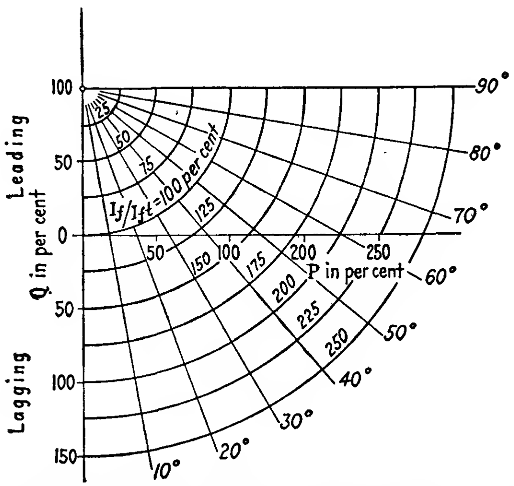

FIG. 183.—Simplified chart for non-salient-pole generator. The circles in this chart represent ratios of actual field current to field current corresponding to terminal voltage as read from the magnetization curve. The chart is applicable at any value of terminal voltage but must be used in conjunction with the magnetization curve. The angle lines indicate displacement between excitation voltage and terminal voltage. The resistance is neglected.

Synchronous Motors and Condensers.—The performance of synchronous motors may be determined by charts of the same general type as those used for the generators. While the charts representing generator action fundamentally are *receiving charts*, the charts for motor action are *sending charts*. The calculation and construction of the motor charts, however, when friction and windage loss and core loss are neglected, involve no principles which have not already been covered in the discussion of the generator charts.[1]

[1] Assuming that the core loss is a function of the terminal voltage, the effect of friction and windage, and core loss can easily be taken into account.

A chart constructed for a generator may, in general, be used to give the performance of the same machine when operating as a motor, provided the induced-voltage circles and the field-current curves are extended into the negative-power region. Operating points for motor action are then located to the left of the vertical or reactive-power axis.

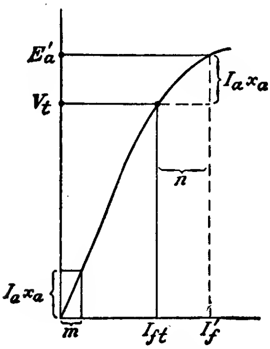

FIG. 184.—Magnetization curve of non-salient-pole generator. The simplified chart utilizes the equivalent field current m for the leakage-reactance drop at all saturations. This value is too low at higher saturations where it appropriately should have the value n.

If the motor resistance is neglected, as was previously done for the generators, the generator charts may directly be used to represent motor action without extension, provided the sign of the reactive power is reversed. Leading reactive power on the chart should, in other words, be considered as lagging and *vice versa*. The correctness of this is obvious when it is recalled that a synchronous motor, when operating overexcited, draws a leading current, while the current is lagging when the machine operates underexcited. If the synchronous motor operates as a condenser and all losses are neglected, its performance may be determined by selecting operating points along the reactive-power axis.

The synchronous condensers used for voltage control in transmission systems are, as a rule, designed to act as condensers alone and are not intended to supply mechanical power to external loads. It is unnecessary, therefore, to prepare complete charts in order to determine the performance of such machines. A family of V curves suffices, as from this the reactive current and reactive kilovolt-amperes may be obtained for any value of field current.

The sum of these losses, then, is constant for each terminal voltage and may be added on to the input power, when the charts are drawn.

It may be more nearly correct, however, to consider the core loss as depending upon the *air-gap voltage*. If a parabolic relation is assumed, the core loss may be taken care of by attaching, in the circuit representing the motor, a constant shunt resistance at the air-gap terminal of the leakage impedance. The value of the shunt resistance may be based on the open-circuit core loss at normal voltage.

The V curves may be obtained by test or by calculation. With large machines, it may be impracticable to obtain the curves by test, and for such machines, therefore, they are almost always determined by calculation. In order to calculate these curves, it is necessary to have the open-circuit characteristic of the machine and to know the leakage reactance and the coefficient k representing the effect of the armature reaction. The leakage reactance may be obtained in the ordinary manner by means of the Potier triangle, when an open-circuit characteristic and a zero-power-factor curve are available. The coefficient k may be obtained either from an open-circuit and a short-circuit characteristic or from an open-circuit and a zero-power-factor characteristic in the manner previously described in connection with the generator charts.

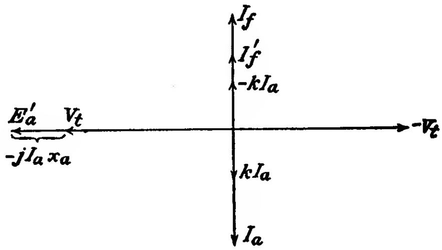

FIG. 185.—Vector diagram of a synchronous condenser operating overexcited.

Figure 185 shows a vector diagram of a synchronous condenser operating overexcited and drawing a leading current. The air-gap voltage (E_a') is obtained from

$$E_a' = V - jI_a x_a \tag{39}$$

and the field current is given by

$$I_f = I_f' - kI_a \tag{40}$$

in which I_f' represents the field current corresponding to the air-gap voltage (E_a'), as obtained from the magnetization curve.

Usually, the V curves are plotted in terms of reactive current *versus* field current. They may also be plotted in terms of reactive power *versus* field current. It is advantageous, as a rule, to use reactive current rather than reactive power, because, when plotted in this manner, the V curves, for all practical purposes, become straight lines. It is sufficient, therefore, to calculate but one point on each slope of the V, the bottom point obviously being obtained directly from the magnetization curve. In carrying through the calculations for a family of V curves, it is most

practical to calculate points corresponding to 100 per cent reactive current, although any arbitrary value may, of course, be used.

Figure 186 shows the characteristics of a large synchronous condenser of modern design. The open-circuit, zero-power-factor and short-circuit saturation curves are given and also a complete family of V curves. The latter covers a voltage range

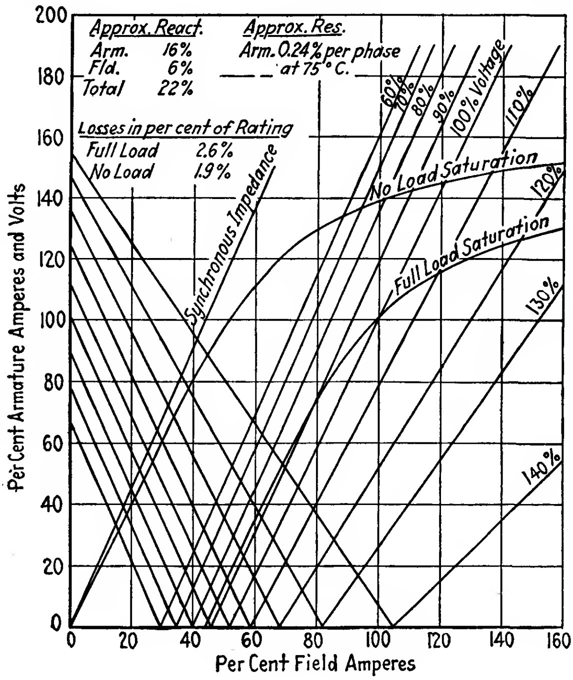

FIG. 186.—Characteristic curves of a large synchronous condenser.

from 60 to 180 per cent of normal. Points on any V curve may easily be checked by calculation, using data from the other characteristic curves.

Large synchronous condensers of modern design usually have the same kilovolt-ampere rating overexcited and underexcited. With smaller condensers, on the other hand, it frequently happens that the rating underexcited is somewhat less than the rating overexcited.[1] This is due to the fact that these machines have

[1] ALGER, P. L., "Synchronous Condensers," *Jour. A.I.E.E.*, p. 1330, December, 1927. In this paper, attention is called to the fact that a more economical design results when the lagging kilovolt-ampere capacity is less

such constants that the field current, in order to give the same rating underexcited as overexcited, would have to be decreased to a prohibitive degree. It might even be necessary to reverse the field current. It is always undesirable to operate a synchronous machine at a very low excitation. In addition, the exciters and the control equipment (Tirrill regulators) are not sufficiently flexible to vary the field current through such an extreme range.

EXAMPLE 1

This example illustrates in detail the calculations of the necessary data for the construction of charts for a salient-pole generator.

Statement of Problem

The characteristic curves of a large salient-pole generator are given in Fig. 187. The ratio of pole arc to pole pitch is ⅔.

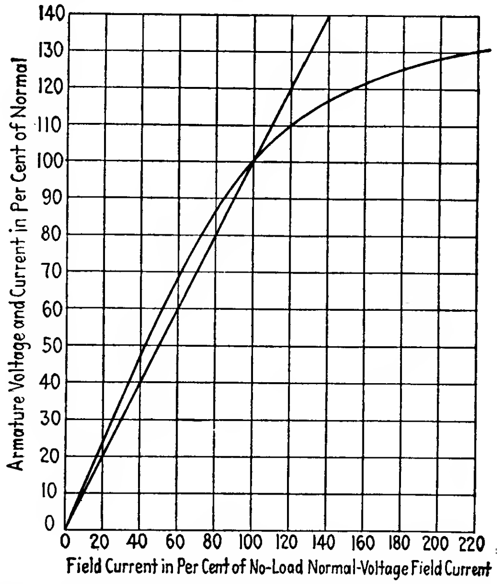

FIG. 187.—Typical saturation and synchronous-impedance curves of a large salient-pole, 60-cycle, 0.8 power-factor generator.

Short-circuit ratio	= 1.0
Armature resistance	= 0.4 per cent.
Armature leakage reactance	= 23 per cent.
Transient reactance	= 30 per cent.

Prepare charts giving the performance of this generator at terminal voltages of 100, 95, 90, 85, 80, and 70 per cent of normal.

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than the leading. It is suggested that the use of static reactors in parallel with the synchronous condenser be considered in cases where a large value of lagging reactive power is required.

Solution

Necessary Generator Constants.

Leakage reactance x_a = 23 per cent

Leakage-reactance drop at rated armature current

$$x_a I_a = 23 \text{ per cent}$$

From the magnetization curve, the field current corresponding to this drop is

$$I_f' = 20 \text{ per cent}$$

The coefficient k representing the effect of the armature reaction is, hence,

$$k = \frac{I_f - I_f'}{I_a} = \frac{100 - 20}{100}100 = 80 \text{ per cent}$$

Demagnetizing coefficient (see equation (28))

$$K_d = 0.45\frac{\sin\frac{a\pi}{b2}}{a\pi/b2} = 0.45\frac{\sin\frac{2\pi}{6}}{2\pi/6} = 0.372$$

Cross-magnetizing coefficient (see equation (32))

$$K_c = \frac{0.9}{\pi}\left[\frac{a}{b} - \frac{1}{\pi}\sin\frac{a\pi}{b}\right]$$

$$= \frac{0.9}{\pi}\left[\frac{2}{3} - \frac{1}{\pi}\sin\frac{2\pi}{3}\right] = 0.112$$

$$\frac{K_c}{K_d} = \frac{0.112}{0.372} = 0.301$$

Circles of induced voltage.

Horizontal displacement of center = 0

Vertical displacement of center $= \dfrac{V_t^2}{x_a}$

Radius of circles $= \dfrac{E_a V_t}{X_a}$.

Table XVIII gives the necessary values calculated from these formulas.

TABLE XVIII.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). CIRCLES OF INDUCED VOLTAGE

V_t, per cent	Vertical displacement of center, per cent	Radii of circles in per cent		
		$E_a' = 200$ per cent	$E_a' = 100$ per cent	$E_a' = 50$ per cent
100	435	870	435	217.5
95	392.5	826	413	206.5
90	352.5	782	391	195.5
85	314.5	739	369.5	184.8
80	278.5	696	348	174
70	213	608.8	304.4	152.2

It will be noted that the radii have been computed for only three values of E_a'. This is fully sufficient, however, since these circles are concentric and equidistant. The radii of intermediate circles are more conveniently obtained by construction when the charts are being drawn.

Auxiliary Chart.

Horizontal displacement of center = 0

Vertical displacement of center $= \dfrac{100}{k} = \dfrac{100}{0.8} = 125$ per cent

Radii of circles $= \dfrac{100}{k}\left(\dfrac{I_f}{I'_f}\right) = \dfrac{100}{0.8}\left(\dfrac{I_f}{I'_f}\right) = 125\left(\dfrac{I_f}{I'_f}\right)$ per cent

The auxiliary chart is shown in Fig. 188

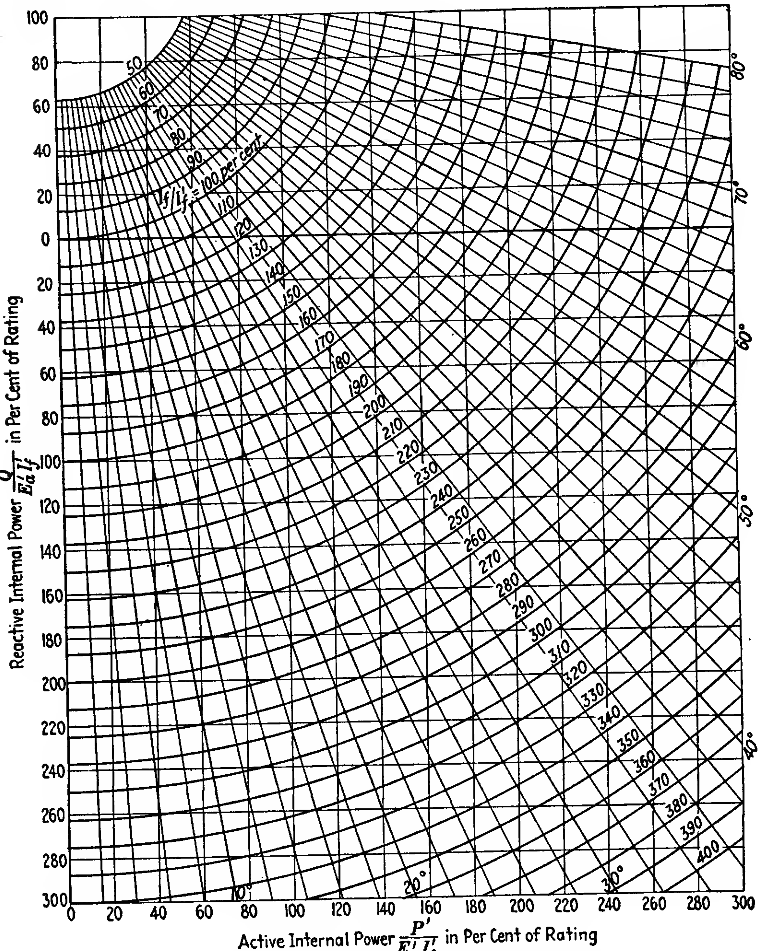

FIG. 188.—Auxiliary chart for salient-pole generator. The characteristic curves are given in Fig. 187 and the calculations in Example 1.

Curves of Constant Field Current.

By using the method previously described for the salient-pole machine and the arrangement suggested in Table XVII, the necessary numerical data are

TABLE XIX.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). FIELD-CURRENT CURVES

V_t, per cent	P, per cent	Q, per cent	$\angle\frac{E'_a}{V_t}$, degrees	E'_a, per cent	I'_f, per cent	P', per cent	Q', per cent	$\frac{P'}{E'_aI'_f}$, per cent	$\frac{Q'}{E'_aI'_f}$, per cent	$\angle\frac{E_a}{E'_a}$, degrees	$\frac{I_f}{I'_f}$, per cent	$\angle\frac{E_a}{V_t}$, degrees	I_f, per cent
100	0	+100	0	77.0	69	0	+ 77.0	0	+145	180	.....	180	
100	0	+ 75	0	82.7	76	0	+ 62.1	0	+ 98.9	0	21.0	0	16.0
100	0	+ 50	0	88.5	83	0	+ 44.3	0	+ 60.2	0	51.5	0	42.8
100	0	+ 25	0	94.3	91	0	+ 23.6	0	+ 27.4	0	77.5	0	70.6
100	0	0	0	100.0	100	0	0	0	0	0	100.0	0	100.0
100	0	− 25	0	105.8	111	0	− 26.4	0	− 22.6	0	117.3	0	130.0
100	0	− 50	0	111.5	124	0	− 55.8	0	− 40.3	0	132.0	0	164.0
100	0	− 75	0	117.2	141	0	− 88.0	0	− 53.2	0	142.0	0	200.0
100	0	−100	0	123.0	166	0	−123.0	0	− 60.2	0	147.5	0	245.0
100	0	−125	0	128.8	202	0	−160.9	0	− 61.7	0	148.5	0	300.0
100	0	−150	0	134.5	.....	0	−201.8	0	....	0	.....	0	
100	25	+100	4.6	77.3	70.0	25	+ 75.6	46.2	+140.0				
100	25	+ 75	4.3	83.0	76.0	25	+ 60.6	39.6	+ 96.2	7.0	27.0	11.3	20.5
100	25	+ 50	4.0	88.7	83.0	25	+ 42.8	34.1	+ 58.3	6.0	54.0	10.0	45.0
100	25	+ 25	3.8	94.3	91.0	25	+ 22.1	29.2	+ 25.7	5.2	82.0	9.0	74.5
100	25	0	3.6	100.1	100.0	25	− 1.4	25.0	− 1.4	4.0	102.0	7.6	102.0
100	25	− 25	3.4	105.9	111.0	25	− 27.9	21.3	− 23.8	3.3	119.0	6.7	132.0
100	25	− 50	3.2	111.7	124.0	25	− 57.2	18.1	− 41.3	2.8	133.0	6.0	165.0
100	25	− 75	3.0	117.5	142.0	25	− 89.4	15.0	− 56.3	2.1	145.0	5.1	206.0
100	25	−100	2.9	123.0	166.0	25	−124.4	12.3	− 60.9	1.8	148.0	4.7	246.0
100	25	−125	2.7	129.0	204.0	25	−162.4	9.5	− 61.8	1.4	149.0	4.1	304.0
100	25	−150	...	.....	.....	25	−203.2						
100	50	+100	9.3	78.0	71.0	50	+ 71.2	90.3	+129.0	17.0	19.0	26.3	13.5
100	50	+ 75	8.6	84.5	78.0	50	+ 56.3	75.9	+ 85.5	13.0	44.0	21.6	34.4
100	50	+ 50	8.0	89.3	84.5	50	+ 38.5	66.3	+ 51.0	10.0	70.0	18.0	59.1
100	50	+ 25	7.6	95.0	91.5	50	+ 17.8	57.5	+ 20.5	8.4	89.0	16.0	81.5
100	50	0	7.2	100.6	101.0	50	− 5.8	49.2	− 5.7	7.0	108.0	14.2	109.0
100	50	− 25	6.8	106.3	112.0	50	− 32.2	42.0	− 27.0	5.8	124.0	12.6	139.0
100	50	− 50	6.4	112.0	125.0	50	− 61.5	35.7	− 44.0	4.9	137.0	11.3	171.0
100	50	− 75	6.1	118.0	144.0	50	− 93.7	29.4	− 55.0	4.0	145.0	10.1	209.0
100	50	−100	5.8	123.3	168.0	50	−128.8	24.1	− 62.0	3.3	150.0	9.1	252.0
100	50	−125	5.5	129.2	206.0	50	−166.7	18.8	− 62.6	2.8	150.0	8.3	309.0
100	50	−150	...	.....	.....	50	−207.5						
100	75	+100	14.0	79.0	72.0	75	+ 64.0	132.0	112.5	24.0	51.0	38.0	36.7
100	75	+ 75	13.0	84.5	78.0	75	+ 49.1	114.0	74.5	18.2	67.0	31.2	52.3
100	75	+ 50	12.0	90.0	85.0	75	+ 31.3	98.5	41.0	15.0	85.0	27.0	72.2
100	75	+ 25	11.5	95.8	93.0	75	+ 10.6	84.5	11.9	12.6	103.0	24.1	95.6
100	75	0	11.0	101.5	102.0	75	− 13.0	72 5	− 12.5	10.2	117.0	21.2	119.0
100	75	− 25	10.2	107.0	113.0	75	− 39.4	62.0	− 32.5	8 5	131.0	18.7	148.0
100	75	− 50	9.7	112.7	127.0	75	− 68.7	52.5	− 48.1	6 8	142.0	16.5	180.0
100	75	− 75	9.2	118.5	146.0	75	−100.9	43.4	− 58.3	5.9	149.0	15.1	218.0
100	75	−100	8.8	124.3	174.0	75	−136.0	34.7	− 63.0	4.8	152.0	13.6	264.0
100	75	−125	8.4	130.0	214.0	75	−173.8	27.1	− 62.5	3.6	151.0	12.0	323.0
100	75	−150	...	.....	.....	75	−211.6						
100	100	+100	18.3	80.3	73.0	100	+ 54.0	170.0	92.2	28.0	87.0	46.3	63.5
100	100	+ 75	17.1	85.8	79.0	100	+ 39.0	147.0	57.7	22.6	94.0	39.7	74.4
100	100	+ 50	16.1	91.4	87.0	100	+ 21.2	126.0	26.5	18.2	106.0	34.3	92.2
100	100	+ 25	15.0	97.0	95.0	100	+ 0.6	109.5	0.7	15.0	118.0	30.0	112.0
100	100	0	14.3	102.3	104.0	100	− 23.0	94.0	− 21.6	12.5	131.0	26.5	136.0
100	100	− 25	13.5	108.0	115.5	100	− 49.4	80.0	− 39.6	10.2	140.0	23.7	162.0
100	100	− 50	12.8	113.7	130.0	100	− 78.8	68.0	− 53.2	8.5	148.0	21.3	192.0

TABLE XIX.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). FIELD-CURRENT CURVES.—(*Continued*)

V_t, per cent	P, per cent	Q, per cent	$\angle\frac{E'_a}{V_t}$, degrees	E'_a, per cent	I'_f, per cent	P', per cent	Q', per cent	$\frac{P'}{E'_a I'_f}$, per cent	$\frac{Q'}{E'_a I'_f}$, per cent	$\angle\frac{E_a}{E'_a}$, degrees	$\frac{I_f}{I'_f}$, per cent	$\angle\frac{E_a}{V_t}$, degrees	I_f, per cent
100	100	− 75	12.2	119.5	150.0	100	−111.0	56.0	− 62.0	7.2	153.0	19.4	229.0
100	100	−100	11.6	125.0	177.0	100	−146.0	45.3	− 66.2	5.5	155.0	17.1	274.0
100	100	−125	11.1	130.7	222.0	100	−184.0	34.5	− 63.5	4.8	152.0	15.9	338.0
100	100	−150	...	.....	.....	100	−225.0						
100	125	+100	22.6	82.1	75.0	125	+ 41.0	203.0	66.7	30.3	122.0	52.9	91.5
100	125	+ 75	21.1	87.5	81.5	125	+ 26.0	175.0	36.4	25.1	123.0	46.2	100.0
100	125	+ 50	20.0	93.0	89.0	125	+ 8.3	151.0	10.0	20.8	129.0	40.8	115.0
100	125	+ 25	18.7	98.5	97.0	125	− 12.4	131.0	− 12.9	17.4	136.0	36.1	132.0
100	125	0	17.7	104.0	107.0	125	− 35.9	112.5	− 32.3	14.4	144.0	32.1	154.0
100	125	− 25	16.8	109.5	119.0	125	− 62.4	96.2	− 47.9	12.1	151.0	28.9	180.0
100	125	− 50	16.0	115.1	134.0	125	− 91.7	81.0	− 59.2	10.2	156.0	26.2	209.0
100	125	− 75	15.2	120.5	154.0	125	−124.0	67.5	− 67.1	8.4	159.0	23.6	245.0
100	125	−100	14.5	126.1	183.0	125	−159.0	54.2	− 69.0	6.7	159.0	21.2	291.0
100	125	−125	13.9	132.0	238.0	125	−197.0	39.8	− 62.7	5.2	152.0	19.1	362.0
100	125	−150	...	.....	.....	125	−237.5						
100	150	+100	26.6	84.3	78.0	150	+ 25.3	228.0	+ 38.5	31.5	153.0	58.1	119.0
100	150	+ 75	25.0	89.7	84.0	150	+ 10.3	199.0	+ 13.7	26.7	151.0	51.7	127.0
100	150	+ 50	23.5	94.9	91.0	150	− 7.5	173.5	− 8.7	23.0	152.0	46.5	138.0
100	150	+ 25	22.2	100.2	100.5	150	− 28.1	149.0	− 27.9	19.0	154.0	41.2	155.0
100	150	0	21.0	105.5	110.0	150	− 51.8	129.0	− 44.7	16.0	158.0	37.0	174.0
100	150	− 25	20.0	111.3	124.0	150	− 78.0	108.5	− 56.5	13.3	161.0	33.3	200.0
100	150	− 50	19.0	116.5	139.0	150	−107.5	92.6	− 66.2	11.2	164.0	30.2	228.0
100	150	− 75	18.0	122.0	161.0	150	−139.7	76.3	− 71.0	9.6	164.0	27.6	264.0
100	150	−100	17.3	127.6	193.0	150	−174.7	60.8	− 70.8	7.9	161.0	25.2	310.0
100	150	−125	16.5	133.0	248.0	150	−212.6	45.4	− 64.5	5.8	154.0	22.3	382.0
100	150	−150	...	.....	.....	150	−253.5						
100	175	+100	31.3	87.3	81.5	175	+ 6.5	246.0	+ 9.1	31.6	182.0	62.9	148.0
100	175	+ 75	29.4	92.4	88.0	175	− 8.5	215.0	− 10.5	27.2	176.0	56.6	150.0
100	175	+ 50	27.7	97.5	95.0	175	− 26.2	189.0	− 28.2	23.5	172.0	51.2	164.0
100	175	+ 25	26.1	102.7	102.0	175	− 47.0	167.0	− 44.9	20.5	173.0	46.6	177.0
100	175	0	25.9	108.0	116.0	175	− 70.5	140.0	− 56.2	16.9	171.0	42.8	198.0
100	175	− 25	23.6	113.5	130.0	175	− 97.0	118.0	− 65.8	14.3	170.0	37.9	221.0
100	175	− 50	22.5	118.7	146.0	175	−126.2	101.0	− 72.9	12.0	171.0	34.5	250.0
100	175	− 75	21.5	124.3	173.0	175	−158.5	81.2	− 73.6	9.8	167.0	31.3	289.0
100	175	−100	20.5	129.7	211.0	175	−193.5	64.0	− 70.6	8.1	161.0	28.6	339.0
100	175	−125	19.7	135.3	.....	175	−231.2						
100	175	−150	...	.....	.....	175	−272.2						
100	200	+100	34.2	89.7	84.0	200	− 15.0	265.0	− 19.9	31.8	210.0	65.0	176.0
100	200	+ 75	32.1	94.5	91.0	200	− 30.0	232.0	− 34.8	27.8	199.0	59.9	181.0
100	200	+ 50	30.4	99.7	99.5	200	− 47.5	202.0	− 47.8	24.0	191.0	54.4	190.0
100	200	+ 25	28.7	104.6	108.0	200	− 68.6	177.0	− 60.7	20.8	188.0	49.5	203.0
100	200	0	27.3	110.0	120.0	200	− 92.0	152.0	− 69.6	18.0	185.0	45.3	222.0
100	200	− 25	26.0	115.1	134.0	200	−118.6	130.0	− 76.7	15.2	183.0	41.2	245.0
100	200	− 50	24.7	120.5	154.0	200	−147.5	108.0	− 79.4	12.7	178.0	37.4	274.0
100	200	− 75	23.6	125.7	181.0	200	−180.0	88.0	− 79.0	10.4	172.0	34.0	312.0
100	200	−100	21.2	130.0	214.0	200	−215.0	71.9	− 77.3	8.8	168.0	30.0	360.0
100	200	−125	....	.....	.....	200	−253.0						
100	200	−150	....	.....	.....	200	−294.0						
90	0	+100	0	64.5	57.0	0	+ 71.6	0	+194.5	180	5.5	180	−31.6
90	0	+ 80	0	69.6	60.9	0	+ 62.0	0	+146.2	180	17.0	180	−10.3
90	0	+ 60	0	74.6	66.9	0	+ 49.8	0	+100.0	0	20.0	0	+13.4

TABLE XIX.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). FIELD-CURRENT CURVES.—(*Continued*)

V_t, per cent	P, per cent	Q, per cent	$\angle\frac{E'_a}{V_t}$, degrees	E'_a, per cent	I'_f, per cent	P', per cent	Q', per cent	$\frac{P'}{E'_a I'_f}$, per cent	$\frac{Q'}{E'_a I'_f}$, per cent	$\angle\frac{E_a}{E'_a}$, degrees	$\frac{I_f}{I'_f}$, per cent	$\angle\frac{E_a}{V_t}$, degrees	I_f, per cent
90	0	+ 40	0	79.8	72.9	0	+ 35.5	0	+ 61.0	0	51.0	0	37.2
90	0	+ 20	0	84.9	79.2	0	+ 18.9	0	+ 28.0	0	76.5	0	60.6
90	0	0	0	90.0	85.0	0	0	0	0	0	100.0	0	85.0
90	0	− 20	0	95.1	92.0	0	− 21.1	0	− 24.0	0	119.5	0	109.5
90	0	− 40	0	100.2	100.3	0	− 44.5	0	− 42.3	0	134.5	0	135.0
90	0	− 60	0	105.3	110.0	0	− 70.2	0	− 60.5	0	149.0	0	164.0
90	0	− 80	0	110.4	121.0	0	− 98.0	0	− 73.0	0	159.0	0	192.5
90	0	−100	0	115.5	135.0	0	−128.4	0	− 82.0	0	164.5	0	222.0
90	0	−120	0	120.6	155.0	0	−160.8	0	− 86.5	0	169.0	0	262.0
90	0	−140	0	125.1	180.0	0	−195.6	0	− 86.6	0	169.0	0	304.0
90	0	−160	0	130.8	221.5	0	−232.5	0	− 80.5	0	.....	0	
90	20	+100	4.6	64.5	56.7	20	+ 70.4	36.0	+192.0				
90	20	+ 80	4.2	69.8	62.0	20	+ 61.0	47.0	+141.0	195.0	6.0	199.2	−3.7
90	20	+ 60	4.0	74.8	67.0	20	+ 48.7	40.3	+ 97.1	7.0	26.5	11.0	17.8
90	20	+ 40	3.7	79.9	73.0	20	+ 34.3	34.3	+ 58.7	6.0	56.5	9.7	41.2
90	20	+ 20	3.5	85.0	78.2	20	+ 17.7	30.0	+ 26.6	4.4	81.0	7.9	63.3
90	20	0	3.3	90.1	86.0	20	− 1.1	25.8	− 1.4	4.0	103.0	7.3	88.5
90	20	− 20	3.1	95.2	92.2	20	− 22.3	22.8	− 26.0	3.0	122.0	6.1	112.7
90	20	− 40	2.9	100.3	100.6	20	− 45.7	20.0	− 45.2	2.4	137.0	5.3	138.0
90	20	− 60	2.8	105.0	110.0	20	− 71.3	17.4	− 61.2	2.2	149.0	5.0	164.0
90	20	− 80	2.7	110.5	122.0	20	− 99.3	15.1	− 73.8	2.0	157.5	4.7	194.0
90	20	−100	2.5	115.6	136.0	20	−129.5	13.0	− 82.2	1.8	166.5	4.3	225.7
90	20	−120	2.4	120.7	155.5	20	−162.1	11.0	− 86.3	1.5	170.0	3.9	264.5
90	20	−140	2.3	125.8	181.8	20	−197.8	9.2	− 86.7	1.3	170.0	3.6	309.0
90	20	−160	2.2	130.9	222.0	20	−245.2	7.5	− 84.4	1.0	168.0	3.2	373.0
90	40	+100	9.0	65.2	57.0	40	+ 67.1	109.1	+181.0	19.4	4.5	28.4	2.6
90	40	+ 80	8.3	70.3	62.3	40	+ 57.2	92.2	+130.2	17.0	18.0	25.3	11.2
90	40	+ 60	7.8	75.3	67.7	40	+ 45.2	78.8	+ 88.4	14.0	40.5	21.8	27.4
90	40	+ 40	7.3	80.4	73.6	40	+ 30.8	69.5	+ 52.1	11.5	67.5	18.8	49.7
90	40	+ 20	6.9	85.4	79.0	40	+ 14.3	59.4	+ 21.2	10.0	92.0	16.9	72.6
90	40	0	6.5	90.5	85.5	40	− 4.5	51.8	− 5.8	8.7	109.0	15.2	93.8
90	40	− 20	6.2	95.6	92.8	40	− 25.7	45.2	− 28.9	5.9	126.0	12.0	117.0
90	40	− 40	5.8	100.7	101.0	40	− 49.2	39.4	− 48.3	4.9	141.0	10.7	142.0
90	40	− 60	5.5	105.8	110.7	40	− 74.8	34.5	− 64.0	4.1	153.0	9.6	169.0
90	40	− 80	5.3	110.9	122.0	40	−102.8	29.0	− 75.5	3.6	162.0	8.9	198.0
90	40	−100	5.1	115.9	136.6	40	−132.9	25.6	− 83.9	3.1	168.0	8.2	228.0
90	40	−120	4.8	121.0	156.2	40	−165.5	21.6	− 87.6	2.7	171.0	7.5	267.0
90	40	−140	4.7	126.1	184.9	40	−200.3	17.6	− 86.0	2.2	169.0	6.9	312.0
90	40	−160	4.5	131.1	226.0	40	−237.3	14.0	− 80.1	2.0	165.0	6.5	373.0
90	60	+100	13.3	66.2	58.0	60	+ 61.4	158.0	+160.0	32.0	42.5	45.3	24.6
90	60	+ 80	12.4	71.1	63.0	60	+ 55.6	134.6	+124.0	25.0	45.0	37.4	28.4
90	60	+ 60	11.6	76.1	68.0	60	+ 35.6	116.6	+ 68.7	19.3	73.0	30.9	46.0
90	60	+ 40	10.9	81.1	74.0	60	+ 25.3	100.2	+ 42.1	15.0	85.0	25.9	63.7
90	60	+ 20	10.2	86.1	80.0	60	+ 8.6	87.3	+ 12.5	12.3	102.5	22.5	82.0
90	60	0	9.7	91.2	86.0	60	− 10.2	76.8	− 13.0	10.2	119.5	19.9	102.6
90	60	− 20	9.2	96.2	93.0	60	− 31.4	67.2	− 35.0	8.7	135.0	17.9	125.4
90	60	− 40	8.7	101.3	101.0	60	− 54.7	59.0	− 53.4	7.4	148.0	16.1	149.5
90	60	− 60	8.3	106.3	111.0	60	− 81.1	51.2	− 68.6	6.2	158.5	14.5	176.0
90	60	− 80	7.9	111.3	123.0	60	−108.4	44.2	− 79.0	5.3	166.0	13.2	204.0
90	60	−100	7.6	116.5	139.0	60	−138.6	37.5	− 85.6	4.6	170.5	12.2	237.0
90	60	−120	7.2	121.5	155.0	60	−171.1	32.3	− 90.8	4.0	174.0	11.2	270.0

TABLE XIX.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). FIELD-CURRENT CURVES.—(*Continued*)

V_t, per cent	P, per cent	Q, per cent	$\angle \frac{E'_a}{V_t}$, degrees	E'_a, per cent	I'_f, per cent	P', per cent	Q', per cent	$\frac{P'}{E'_a I'_f}$, per cent	$\frac{Q'}{E'_a I_f}$, per cent	$\angle \frac{E_a}{E'_a}$, degrees	$\frac{I_f}{I'_f}$, per cent	$\angle \frac{E_a}{V_t}$, degrees	I_f, per cent
90	60	−140	6.9	126.6	187.0	60	−206.0	25.8	− 87.1	3.4	171.0	10.3	320.0
90	60	−160	6.7	131.6	230.0	60	−243.0	20.3	− 80.4	2.6	165.5	9.3	381.0
90	80	+100	17.6	67.6	59.0	80	+ 53.5	202.0	+134.0	35.6	88.0	53.2	51.9
90	80	+ 80	16.4	72.3	64.0	80	+ 43.6	174.0	+ 94.4	28.9	87.7	45.1	56.1
90	80	+ 60	15.3	77.3	70.0	80	+ 31.6	148.5	+ 58.2	22.5	94.5	37.9	66.1
90	80	+ 40	14.4	82.2	75.0	80	+ 17.2	130.0	+ 27.9	8.9	106.7	32.3	79.9
90	80	+ 20	13.5	87.1	81.0	80	+ 1.0	113.2	+ 1.4	5.6	119.0	29.1	96.4
90	80	0	12.8	92.2	87.0	80	− 18.2	100.0	− 22.7	13.0	133.5	25.8	116.2
90	80	− 20	12.2	97.1	95.0	80	− 39.0	87.0	− 42.3	10.5	144.5	22.7	137.3
90	80	− 40	11.6	102.2	104.0	80	− 62.8	75.6	− 58.3	8.8	154.0	20.4	160.1
90	80	− 60	11.0	107.1	113.0	80	− 88.0	66.5	− 72.6	6.9	162.5	17.9	183.5
90	80	− 80	10.5	112.2	125.0	80	−116.4	57.6	− 82.9	6.8	170.5	17.3	213.0
90	80	−100	10.1	117.3	142.0	80	−146.5	48.6	− 88.0	5.6	173.5	15.7	246.5
90	80	−120	9.7	122.3	163.0	80	−179.0	40.6	− 89.9	5.0	174.0	14.7	283.5
90	80	−140	9.3	127.2	191.0	80	−214.0	38.4	− 88.0	4.0	172.0	13.3	328.5
90	80	−160	8.9	132.4	240.0	80	−251.0	25.6	− 79.0	3.2	164.0	12.1	394.0
90	100	+100	21.7	69.1	61.0	100	+ 43.2	240.0	+102.0	37.6	131.0	59.3	79.9
90	100	+ 80	20.2	74.1	66.0	100	+ 33.4	206.0	+ 68.2	30.7	123.0	50.9	81.2
90	100	+ 60	18.9	78.8	71.0	100	+ 21.4	179.7	+ 38.2	25.8	125.0	44.7	88.7
90	100	+ 40	17.8	83.6	77.0	100	+ 7.0	156.0	+ 10.9	21.5	130.5	39.3	100.5
90	100	+ 20	16.8	88.6	83.0	100	− 9.6	136.0	− 13.0	17.8	138.0	34.6	114.7
90	100	0	15.8	93.5	89.0	100	− 28.4	120.5	− 34.0	15.0	148.5	30.8	132.2
90	100	− 20	15.1	98.4	97.0	100	− 49.6	105.5	− 52.0	12.0	157.5	29.9	152.8
90	100	− 40	14.3	103.2	105.0	100	− 73.0	92.6	− 67.3	11.0	165.0	25.3	173.2
90	100	− 60	13.6	108.2	116.0	100	− 98.0	80.0	− 78.0	9.5	170.5	23.1	198.0
90	100	− 80	13.0	113.2	128.0	100	−126.6	69.5	− 87.4	8.1	176.5	21.1	226.0
90	100	−100	12.5	118.2	145.0	100	−156.8	59.0	− 91.5	6.9	178.0	19.4	258.0
90	100	−120	11.9	123.1	167.0	100	−188.4	49.3	− 91.6	6.0	176.0	17.9	294.0
90	100	−140	11.5	128.2	198.0	100	−224.0	40.0	− 88.3	4.9	173.0	16.4	342.5
90	100	−160	11.0	133.1	247.0	100	−261.0	30.9	− 79.1	4.9	165.0	14.9	409.0
90	120	+100	25.5	71.2	63.0	120	+ 31.7	270.0	+ 70.5	38.4	167.8	63.9	105.6
90	120	+ 80	23.8	75.9	68.0	120	+ 21.0	236.0	+ 40.6	32.5	158.0	56.3	107.3
90	120	+ 60	22.4	80.5	73.0	120	+ 8.9	206.0	+ 15.9	27.5	153.0	49.9	111.6
90	120	+ 40	21.0	85.4	79.0	120	− 5.5	179.0	− 8.2	23.2	154.0	44.2	121.6
90	120	+ 20	19.9	90.1	85.0	120	− 22.0	157.0	− 28.7	19.9	158.0	39.6	134.2
90	120	0	18.8	95.0	92.0	120	− 41.0	138.0	− 46.3	17.0	163.0	38.8	150.0
90	120	− 20	17.9	99.9	99.8	120	− 62.0	121.0	− 62.3	14.6	169.0	32.5	168.8
90	120	− 40	17.0	104.7	109.0	120	− 85.5	106.0	− 75.0	12.5	174.0	29.5	189.7
90	120	− 60	16.3	109.6	119.0	120	−111.1	92.5	− 85.2	10.7	179.0	27.0	213.0
90	120	− 80	15.5	114.5	133.0	120	−139.0	79.5	− 92.0	9.2	181.0	24.7	241.5
90	120	−100	14.8	119.4	150.0	120	−168.3	67.7	− 94.0	7.9	180.5	22.7	271.0
90	120	−120	14.3	124.3	174.0	120	−201.7	56.0	− 94.0	6.7	179.2	21.0	312.0
90	120	−140	13.7	129.3	207.0	120	−236.5	45.4	− 88.5	5.4	173.5	19.1	359.5
90	120	−160	13.2	134.2	255.0	120	−270.4	35.5	− 80.0	4.7	166.0	17.9	424.0
90	140	+100	29.0	73.5	65.0	140	+ 16.0	296.0	+ 33.5	38.0	203.0	67.0	132.0
90	140	+ 80	27.3	78.1	71.0	140	+ 6.1	255.0	+ 11.0	33.6	192.0	60.9	136.2
90	140	+ 60	25.6	82.6	76.0	140	− 6.0	227.0	− 9.6	28.3	182.5	53.9	137.1
90	140	+ 40	24.2	87.2	81.0	140	− 20.2	200.0	− 28.5	24.6	178.0	48.8	144.2
90	140	+ 20	22.9	92.0	87.0	140	− 36.8	176.0	− 46.0	21.2	178.0	44.1	154.9
90	140	0	21.7	96.6	94.0	140	− 55.6	155.0	− 61.2	18.3	180.0	30.0	169.2
90	140	− 20	20.6	101.5	103.0	140	− 76.8	135.0	− 74.0	15.7	182.0	36.3	187.5

TABLE XIX.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). FIELD-CURRENT CURVES.—(*Continued*)

V_t, per cent	P, per cent	Q, per cent	$\angle\frac{E'_a}{V_t}$, degrees	E'_a, per cent	I'_f, per cent	P', per cent	Q', per cent	$\frac{P'}{E'_a I'_f}$, per cent	$\frac{Q'}{E'_a I'_f}$, per cent	$\angle\frac{E_a}{E'_a}$, degrees	$\frac{I_f}{I'_f}$, per cent	$\angle\frac{E_a}{V_t}$, degrees	I_f, per cent
90	140	− 40	19.7	106.2	111.0	140	−100.2	119.8	− 85.0	13.7	185.5	33.4	206.0
90	140	− 60	18.8	111.1	123.0	140	−126.0	103.0	− 92.1	11.8	186.0	30.6	229.5
90	140	− 80	17.9	116.0	137.0	140	−153.9	89.0	− 96.5	10.2	187.0	28.1	258.0
90	140	−100	17.2	120.8	155.0	140	−184.0	75.5	− 98.4	8.7	186.0	25.9	288.5
90	140	−120	16.5	125.6	180.0	140	−216.5	62.7	− 96.0	7.2	182.5	23.7	329.0
90	140	−140	15.9	130.6	220.0	140	−251.0	47.4	− 87.5	5.6	173.0	21.5	381.0
90	140	−160	15.3	135.5	.....	140	−288.0						
90	160	+100	32.5	76.1	68.0	160	− 1.0	312.0	− 1.9	36.2	224.0	68.7	152.2
90	160	+ 80	30.5	80.5	74.0	160	− 11.0	272.0	− 18.5	32.6	214.0	63.1	158.3
90	160	+ 60	28.8	85.0	79.0	160	− 23.0	241.0	− 34.3	28.5	204.0	57.2	161.2
90	160	+ 40	27.1	89.5	84.0	160	− 37.3	215.0	− 49.6	25.1	199.0	52.2	167.4
90	160	+ 20	25.7	94.0	90.0	160	− 53.8	191.0	− 63.6	22.0	197.0	47.7	177.3
90	160	0	24.5	98.8	98.0	160	− 72.6	167.0	− 75.2	14.2	191.0	43.7	187.0
90	160	− 20	23.3	103.3	103.0	160	− 93.8	152.0	− 87.7	17.2	198.0	40.5	202.0
90	160	− 40	22.2	108.2	116.0	160	−117.3	128.5	− 93.5	14.4	194.7	36.6	225.8
90	160	− 60	21.3	112.8	128.0	160	−143.0	113.0	− 99.0	12.6	194.0	33.9	249.0
90	160	− 80	20.4	117.6	142.0	160	−171.0	97.0	−102.5	10.8	193.0	31.2	274.0
90	160	−100	19.5	122.3	162.0	160	−201.0	81.4	−102.0	9.2	189.5	28.7	307.0
90	160	−120	18.8	127.2	190.0	160	−233.6	66.8	− 96.5	7.7	183.0	26.5	347.5
90	160	−140	18.0	132.0	237.0	160	−268.3	52.0	− 86.0	6.2	172.0	24.2	408.0
90	160	−160	17.4	137.0	.....	160	−305.0						
90	180	+100	35.5	79.0	72.0	180	− 20.0	322.0	− 35.1	35.9	254.0	71.4	183.0
90	180	+ 80	33.5	83.2	77.0	180	− 30.0	283.0	− 46.8	31.9	236.0	65.3	182.0
90	180	+ 60	32.7	87.5	82.0	180	− 42.0	253.0	− 58.5	28.5	225.0	61.2	184.0
90	180	+ 40	30.0	92.0	87.0	180	− 56.5	227.0	− 70.5	25.3	218.5	55.3	190.0
90	180	+ 20	28.5	96.4	94.0	180	− 73.0	200.0	− 80.6	22.2	212.5	50.7	199.6
90	180	0	27.1	100.9	101.0	180	− 92.0	178.0	− 90.2	19.5	209.2	46.6	211.0
90	180	− 20	25.8	105.5	111.0	180	−113.0	155.0	− 96.5	17.1	205.5	42.9	228.0
90	180	− 40	24.7	110.1	121.0	180	−136.5	136.5	−102.2	15.0	204.5	39.7	247.0
90	180	− 60	23.6	114.8	133.0	180	−162.0	119.0	−106.0	13.0	202.0	36.6	269.0
90	180	− 80	22.6	119.5	150.0	180	−190.0	101.2	−106.0	11.1	197.0	33.7	295.0
90	180	−100	21.7	124.1	172.0	180	−222.0	85.5	−103.0	9.8	191.5	31.5	330.0
90	180	−120	20.9	129.0	205.0	180	−253.0	68.6	− 96.0	8.0	183.0	28.9	376.0
90	180	−140	20.1	133.6	247.0	180	−288.0	55.5	− 87.4	6.5	173.5	26.6	429.0
90	180	−160	19.4	138.0	.....	180	−325.0						
90	200	+100	34.4	82.0	75.0	200	− 42.0	328.0	− 68.2	35.0	269.0	69.4	201.0
90	200	+ 80	36.3	86.1	80.0	200	− 52.0	294.0	− 75.3	31.3	259.0	67.6	207.0
90	200	+ 60	34.4	90.4	85.5	200	− 64.0	261.0	− 83.0	27.9	245.0	62.3	209.0
90	200	+ 40	32.6	94.6	91.0	200	− 78.0	234.0	− 90.5	25.0	236.0	57.6	215.0
90	200	+ 20	31.1	99.0	98.9	200	− 92.8	207.0	− 95.0	22.3	225.5	53.4	222.0
90	200	0	29.6	103.3	106.0	200	−113.5	182.0	−103.5	19.5	220.5	49.1	236.0
90	200	− 20	28.2	107.8	115.0	200	−134.8	163.0	−108.9	17.4	217.5	45.6	250.0
90	200	− 40	27.0	112.3	128.1	200	−158.0	141.0	−110.0	15.3	211.0	42.3	270.0
90	200	− 60	25.9	117.0	144.2	200	−184.0	120.0	−110.0	13.4	205.5	39.3	287.0
90	200	− 80	24.8	121.5	159.0	200	−212.0	105.0	−109.5	11.6	201.0	36.4	320.0
90	200	−100	23.8	126.1	185.0	200	−242.0	86.8	−104.0	9.2	191.0	33.0	354.0
90	200	−120	22.9	131.0	222.0	200	−274.0	70.0	− 94.5	8.0	182.0	30.9	
90	200	−140	22.1	135.6	.....	200	−309.0						
90	200	−160	21.3	140.3	.....	200	−346.0						
80	0	+100	0	51.4	44.5	0	+ 64.0	0	+280.0	180.0			
80	0	+ 75	0	58.3	50.5	0	+ 54.8	0	+186.4	180.0			

TABLE XIX.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). FIELD-CURRENT CURVES.—(*Continued*)

V_t, per cent	P, per cent	Q, per cent	$\angle\frac{E'_a}{V_t}$, degrees	E'_a, per cent	I'_f, per cent	P', per cent	Q', per cent	$\frac{P'}{E'_aI'_f}$, per cent	$\frac{Q'}{E'_aI'_f}$, per cent	$\angle\frac{E_a}{E'_a}$, degrees	$\frac{I_f}{I'_f}$, per cent	$\angle\frac{E_a}{V_t}$, degrees	I_f, per cent
80	0	+ 50	0	65.8	57.5	0	+ 41.0	0	+180.4	180.0			
80	0	+ 25	0	72.7	64.0	0	+ 22.8	0	+ 48.9	0			
80	0	0	0	80.0	73.0	0	0	0	0	0			
80	0	− 25	0	87.1	81.0	0	− 27.3	0	− 39.6	0	132.0	0	107.0
80	0	− 50	0	94.2	90.5	0	− 60.0	0	−69.3	0	155.0	0	140.0
80	0	− 75	0	101.6	103.0	0	− 95.2	0	− 90.9	0	173.0	0	178.5
80	0	−100	0	108.5	117.0	0	−136.0	0	−107.1	0	186.0	0	217.5
80	0	−125	0	115.8	137.0	0	−181.1	0	−114.0	0	191.0	0	261.5
80	0	−150	0	122.8	165.0	0	−230.8	0	−113.8	0	191.0	0	315.0
80	25	+100	8.0	51.7	45.0	25	+ 61.8	107.4	+265.5	35.5	.....	43.5	
80	25	+ 75	7.0	58.6	51.0	25	+ 52.6	83.7	+176.0	17.5	19.0	24.5	9.7
80	25	+ 50	6.5	66.0	57.5	25	+ 28.8	65.9	+ 75.8	12.0	40.0	18.5	23.0
80	25	+ 25	6.2	73.0	64.5	25	+ 20.5	53.1	+ 43.6	8.0	71.0	14.3	45.8
80	25	0	6.0	80.2	73.0	25	− 2.2	42.7	− 3.8	6.0	105.0	12.0	76.7
80	25	− 25	4.8	87.4	81.5	25	− 29.5	39.6	− 46.7	5.0	140.0	9.8	114.1
80	25	− 50	4.5	94.5	91.0	25	− 61.2	29.1	− 71.2	3.0	158.0	7.5	143.8
80	25	− 75	4.0	101.8	103.0	25	− 97.4	23.8	− 92.8	2.7	175.0	6.7	180.4
80	25	−100	3.7	108.7	117.0	25	−138.2	19.6	−108.7	2.3	188.0	6.0	220.0
80	25	−125	3.5	116.1	137.0	25	−183.4	15.7	−115.3	1.8	193.0	5.3	264.5
80	25	−150	3.0	123.0	166.0	25	−133.2	12.3	−114.0	1.6	192.0	4.6	318.6
80	50	+100	16.0	53.1	45.5	50	+ 55.0	206.8	+226.8				
80	50	+ 75	13.5	60.0	52.5	50	+ 45.8	158.5	+145.4	29.0	49.0	45.5	25.7
80	50	+ 50	12.2	67.2	59.0	50	+ 32.0	126.2	+ 80.8	21.8	63.0	34.0	37.2
80	50	+ 25	11.2	74.1	66.5	50	+ 13.8	101.5	+ 28.0	15.0	95.0	26.8	63.2
80	50	0	10.2	81.4	74.5	50	− 9.0	82.4	− 14.8	11.3	121.5	21.5	90.6
80	50	− 25	9.2	88.3	83.0	50	− 36.2	68.2	− 49.4	8.5	145.5	17.7	120.8
80	50	− 50	8.5	95.5	92.5	50	− 68.0	56.6	− 76.9	7.0	165.0	15.5	152.8
80	50	− 75	8.0	102.5	104.5	50	−104.2	46.6	− 97.3	5.0	181.0	13.0	189.0
80	50	−100	7.1	109.5	119.0	50	−145.0	38.4	−111.2	4.5	191.0	11.6	227.0
80	50	−125	6.7	116.8	114.0	50	−190.1	30.6	−116.4	3.4	194.3	10.1	271.8
80	50	−150	6.2	123.8	117.0	50	−239.8	23.8	−114.0	2.5	192.2	8.7	327.0
80	75	+100	23.0	55.6	85.5	75	+ 43.8	278.0	+163.2				
80	75	+ 75	20.5	62.3	54.5	75	+ 34.5	221.0	+101.5	35.0	116.0	55.5	63.2
80	75	+ 50	18.0	69.1	61.0	75	+ 20.8	178.0	+ 49.3	26.0	115.5	44.0	70.4
80	75	+ 25	16.5	76.0	68.0	75	+ 2.6	145.2	+ 5.0	19.5	129.0	36.0	87.6
80	75	0	15.0	82.9	75.0	75	− 20.2	120.5	− 32.6	15.0	146.5	30.0	109.4
80	75	− 25	14.0	89.8	89.5	75	− 47.4	93.5	− 59.2	12.0	158.5	26.0	141.9
80	75	− 50	13.0	96.8	94.5	75	− 79.2	81.9	− 86.7	9.5	178.0	22.5	168.2
80	75	− 75	12.0	103.8	106.5	75	−115.5	67.8	−104.5	7.3	188.5	19.3	200.9
80	75	−100	11.0	110.8	122.5	75	−156.2	55 2	−115.0	6.0	195.5	17.0	239.3
80	75	−125	10.5	117.8	143.5	75	−201.4	44.3	−119.1	4.8	197.5	15.3	283.2
80	75	−150	10.0	124.8	176.0	75	−251.0	34.1	−114.5	4.0	192.5	14.0	339.0
80	100	+100	29.0	58.8	52.0	100	+ 28.1	326.5	+ 91.8	45.5	204.0	74.5	106.0
80	100	+ 75	26.5	65.3	58.0	100	+ 18.8	264.0	+ 49.6	36.0	171.5	62.5	99.5
80	100	+ 50	24.0	71.6	63.5	100	+ 5.1	220.0	+ 11.2	27.3	164.0	51.3	104.0
80	100	+ 25	22.0	78.1	71.0	100	− 13.2	180.5	− 23.8	22.5	164.5	44.5	116.6
80	100	0	19.8	85.0	78.5	100	− 36.0	150.0	− 53.9	17.8	172.5	37.6	135.2
80	100	− 25	18.0	91.8	87.5	100	− 63.2	124.5	− 78.5	14.4	182.0	32.4	159.2
80	100	− 50	17.0	98.4	97.0	100	− 94.9	104.7	− 99.4	11.8	192.0	28.8	186.1
80	100	− 75	16.5	105.5	110.5	100	−131.2	85.7	−112.5	9.5	198.0	26.0	218.0
80	100	−100	14.8	112.4	127.0	100	−171.9	70.0	−120.3	7.3	201.8	22.1	256.0

TABLE XIX.—CALCULATION OF GENERATOR CHART (EXAMPLE 1). FIELD-CURRENT CURVES.—(*Continued*)

V_t, per cent	P, per cent	Q, per cent	$\angle\frac{E'_a}{V_t}$, degrees	E'_a, per cent	I'_f, per cent	P', per cent	Q', per cent	$\frac{P'}{E'_aI'_f}$, per cent	$\frac{Q'}{E'_aI'_f}$, per cent	$\angle\frac{E_a}{E'_a}$, degrees	$\frac{I_f}{I'_f}$, per cent	$\angle\frac{E_a}{V_t}$, degrees	I_f, per cent
80	100	−125	14.0	119.2	149.0	100	−217.0	56.4	−123.2	6.2	201.5	20.2	301.0
80	100	−150	13.2	126.3	187.0	100	−267.0	42.3	−111.9	4.8	191.5	18.0	358.0
80	125	+100	35.0	62.5	55.0	125	+ 8.0	364.0	+ 23.3	41.8	265.0	76.8	146.0
80	125	+ 75	32.0	68.5	60.5	125	− 1.3	301.0	− 3.1	36.0	224.0	68.0	135.2
80	125	+ 50	28.0	74.8	67.0	125	− 15.1	249.0	− 30.1	29.0	206.0	57.0	138.0
80	125	+ 25	26.5	83.1	74.5	125	− 33.3	207.0	− 55.0	24.0	198.0	50.5	147.2
80	125	0	24.0	87.6	81.5	125	− 56.1	175.0	− 78.6	19.8	200.0	43.8	163.0
80	125	− 25	22.5	94.3	91.0	125	− 83.3	145.5	− 97.0	16.0	202.0	38.5	184.4
80	125	− 50	21.0	101.0	101.5	125	−115.1	122.0	−112.5	13.5	207.0	34.5	210 0
80	125	− 75	19.5	107.8	115.0	125	−151.3	101.0	−122.5	10.8	209.0	30.3	240.2
80	125	−100	18.2	114.3	132.5	125	−192.0	82.5	−126.5	8.8	208.0	27.0	276.0
80	125	−125	17.0	121.4	158.5	125	−237.2	65.0	−123.2	7.3	203.0	24.3	322.0
80	125	−150	16.2	127.9	195.0	125	−287.0	51.0	−115.1	5.7	195.0	21.9	380.0
80	150	+100	40.0	67.3	59.0	150	− 16.8	378.0	− 42.3	39.3	244.5	79.3	173.5
80	150	+ 75	36.5	72.7	64.5	150	− 26.0	320.0	− 55.4	34.2	262.5	70.7	169.5
80	150	+ 50	33.5	78.5	71.0	150	− 39.8	269.0	− 71.5	28.8	241.7	62.3	171.5
80	150	+ 25	31.0	84.6	78.0	150	− 58.1	227.0	− 88.2	24.2	230.0	55.2	179.5
80	150	0	28.0	91.0	86.0	150	− 80.8	192.0	−103.0	20.5	223.8	48.5	192.6
80	150	− 25	26.5	97.2	95.0	150	−108.1	162.5	−117.2	17.0	223.0	43.5	212.0
80	150	− 50	24.5	103.6	106.0	150	−139.8	136.5	−127.5	14.2	222.5	38.7	236.0
80	150	− 75	23.0	110.2	120.0	150	−176.0	113.5	−133.0	11.7	220.0	34.7	264.0
80	150	−100	21.5	117.0	140.5	150	−216.8	91.1	−132.0	9.3	215.0	30.8	302.0
80	150	−125	20.3	123.4	168.0	150	−262.0	72.2	−126.1	7.6	206.0	28.1	346.0
80	150	−150	19.0	130.4	218.0	150	−311.8	50.0	−104.0	5.8	186.0	24.8	405.0

obtained for the curves of constant field current, and the curves of constant angular displacement between excitation voltage and terminal voltage. Table XIX contains the data for terminal voltages of 100, 90, and 80 per cent. The figures for the other values of terminal voltage are not included.

Auxiliary Curves.

Figures 189 to 200 inclusive show the auxiliary curves of field current and angular displacement from which the desired integral values may be obtained. In order to obtain a maximum number of such values in any region, the curves have been plotted both *versus* reactive power for constant active power and *versus* active power for constant reactive power. The data for these curves are taken from Table XIX.

Main Generator Charts.

The complete charts are shown in Figs. 201 to 206 inclusive.

The circles of constant induced voltage are drawn by using the figures in Table XVIII. The radial, equiangular lines indicating phase displacement between the induced voltage and the terminal voltage are located by means of a protractor.

Points on curves of constant field current and constant displacement beween excitation voltage and terminal voltage are read from the auxiliary curve sheets and laid off in the main charts. Smooth curves are then drawn through the points.

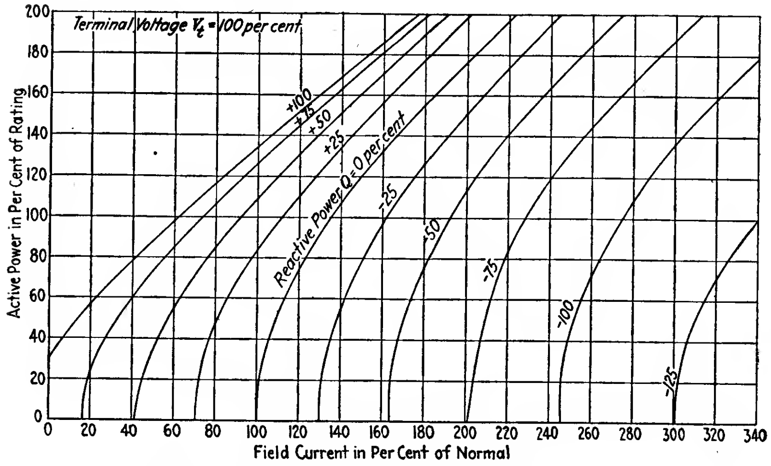

FIG. 189.—Auxiliary field-current curves for salient-pole generator at 100 per cent terminal voltage. These curves are plotted *versus* active power for constant reactive power from data in Table XIX. From these curves integral values of field current have been selected for location in the main generator chart at 100 per cent terminal voltage (Fig. 201).

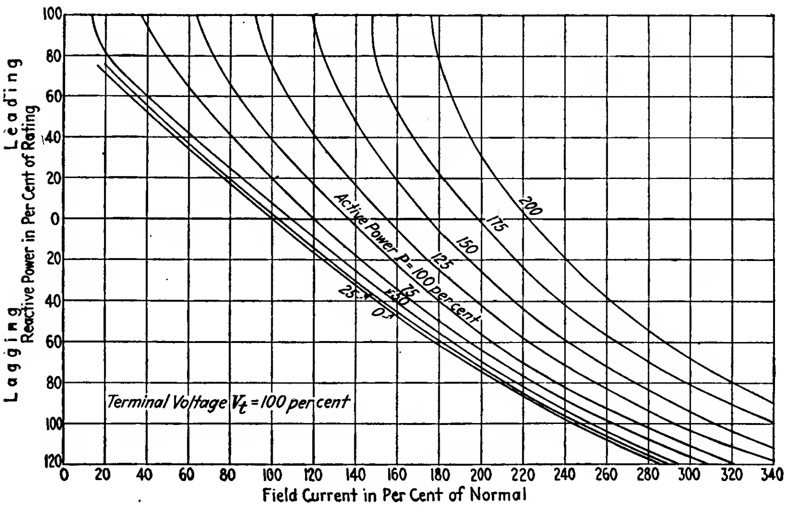

FIG. 190.—Auxiliary field-current curves for salient-pole generator at 100 per cent terminal voltage. These curves are plotted *versus* reactive power for constant active power from data in Table XIX. From these curves integral values of field current have been selected for location in the main generator chart at 100 per cent terminal voltage (Fig. 201).

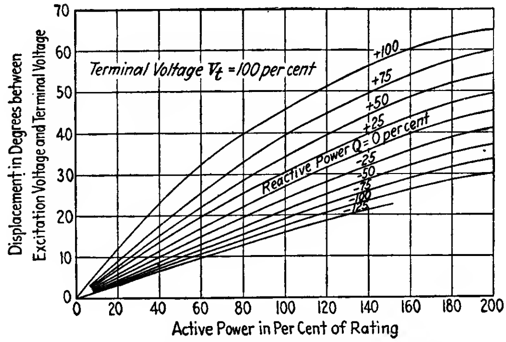

FIG. 191.—Auxiliary angular-displacement curves for salient-pole generator at 100 per cent terminal voltage. These curves are plotted *versus* active power for constant reactive power from data in Table XIX. From these curves integral values of angular displacement between excitation voltage and terminal voltage have been selected for location in the main generator chart at 100 per cent terminal voltage (Fig. 201).

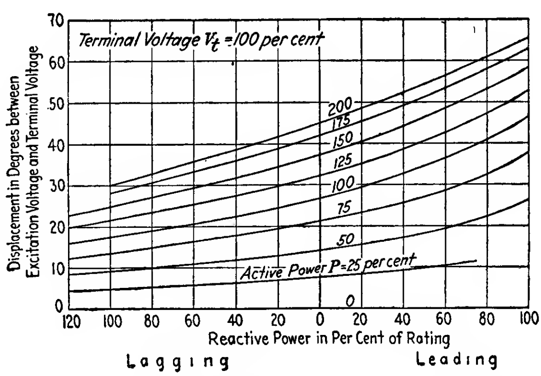

FIG. 192.—Auxiliary angular-displacement curves for salient-pole generator at 100 per cent terminal voltage. These curves are plotted *versus* reactive power for constant active power from data in Table XIX. From these curves integral values of angular displacement between excitation voltage and terminal voltage have been selected for location in the main generator chart at 100 per cent terminal voltage (Fig. 201).

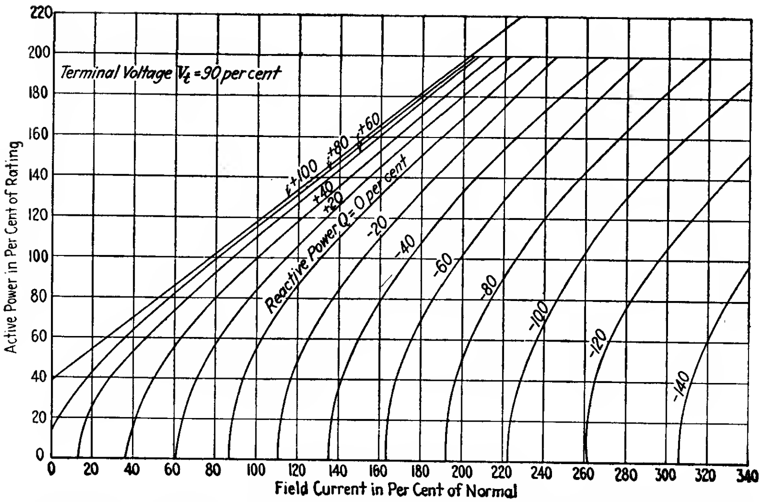

FIG. 193.—Auxiliary field-current curves for salient-pole generator at 90 per cent terminal voltage. These curves are plotted *versus* active power for constant reactive power from data in Table XIX. From these curves integral values of field current have been selected for location in the main generator chart at 90 per cent terminal voltage (Fig. 203).

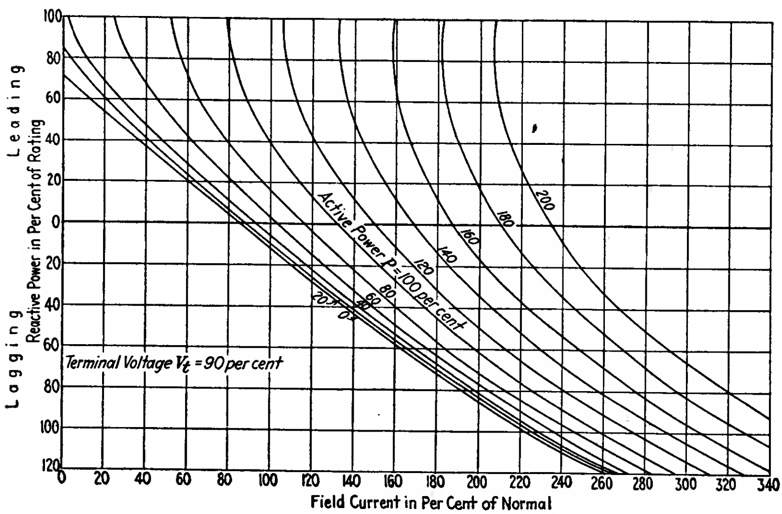

FIG. 194.—Auxiliary field-current curves for salient-pole generator at 90 per cent terminal voltage. These curves are plotted *versus* reactive power for constant active power from data in Table XIX. From these curves integral values of field current have been selected for location in the main generator chart at 90 per cent terminal voltage (Fig. 203).

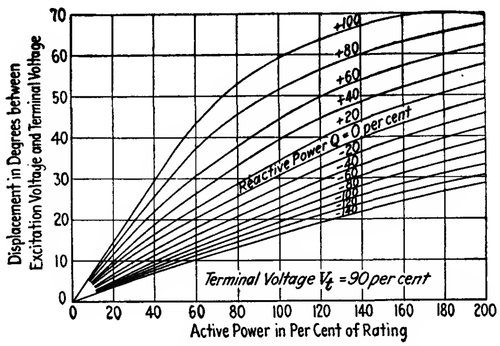

FIG. 195.—Auxiliary angular-displacement curves for salient-pole generator at 90 per cent terminal voltage. These curves are plotted *versus* active power for constant reactive power from data in Table XIX. From these curves integral values of angular displacement between excitation voltage and terminal voltage have been selected for location in the main generator chart at 90 per cent terminal voltage (Fig. 203).

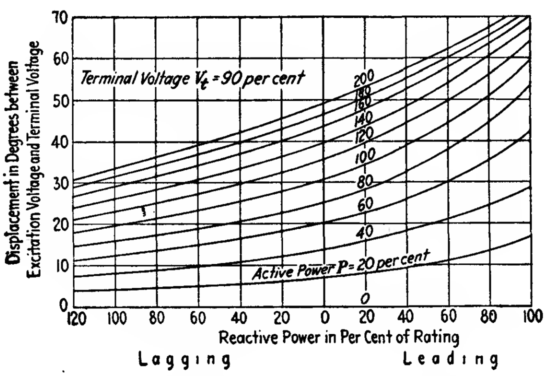

FIG. 196.—Auxiliary angular-displacement curves for salient-pole generator at 90 per cent terminal voltage. These curves are plotted *versus* reactive power for constant active power from data in Table XIX. From these curves integral values of angular displacement between excitation voltage and terminal voltage have been selected for location in the main generator chart at 90 per cent terminal voltage (Fig. 203).

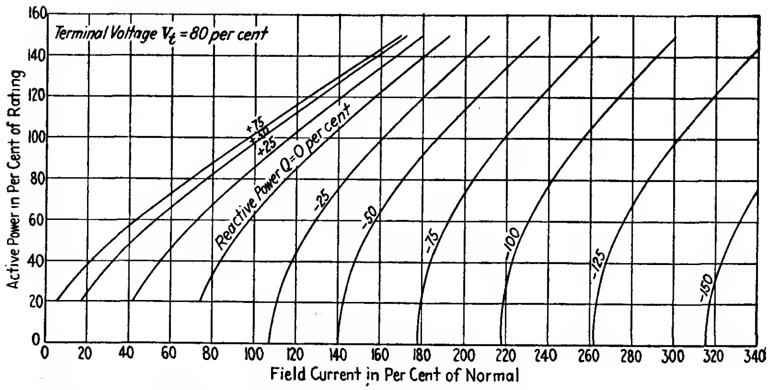

FIG. 197.—Auxiliary field-current curves for salient-pole generator at 80 per cent terminal voltage. These curves are plotted *versus* active power for constant reactive power from data in Table XIX. From these curves integral values of field current have been selected for location in the main generator chart at 80 per cent terminal voltage (Fig. 205).

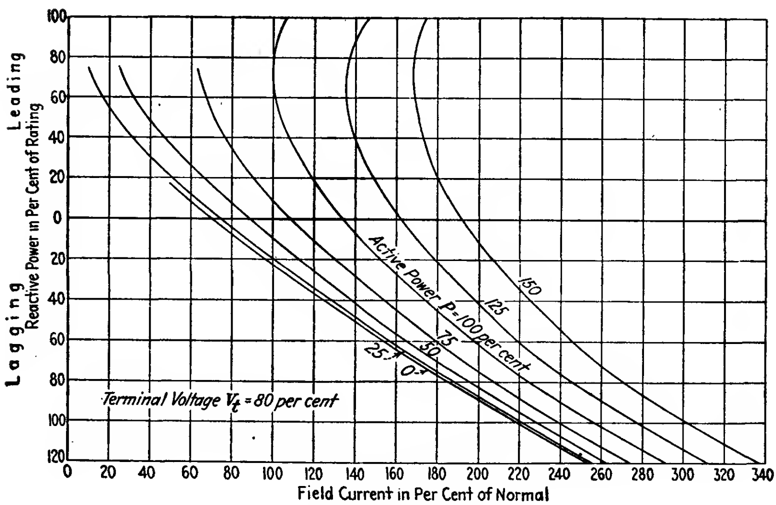

FIG. 198.—Auxiliary field-current curves for salient-pole generator at 80 per cent terminal voltage. These curves are plotted *versus* reactive power for constant active power from data in Table XIX. From these curves integral values of field current have been selected for location in the main generator chart at 80 per cent terminal voltage (Fig. 205).

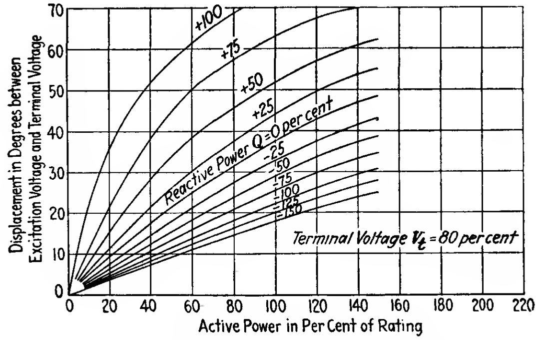

FIG. 199.—Auxiliary angular-displacement curves for salient-pole generator at 80 per cent terminal voltage. These curves are plotted *versus* active power for constant reactive power from data in Table XIX. From these curves integral values of angular displacement between excitation voltage and terminal voltage have been selected for location in the main generator chart at 80 per cent terminal voltage (Fig. 205).

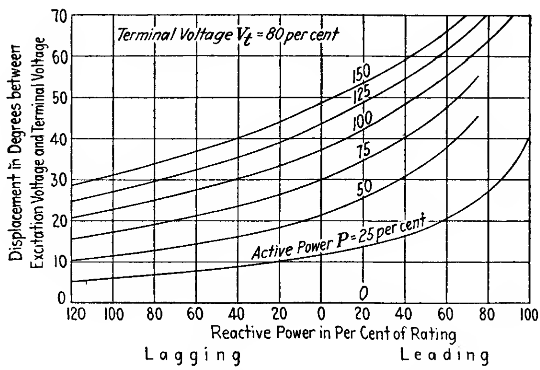

FIG. 200.—Auxiliary angular-displacement curves for salient-pole generator at 80 per cent terminal voltage. These curves are plotted *versus* reactive power for constant active power from data in Table XIX. From these curves integral values of angular displacement between excitation voltage and terminal voltage have been selected for location in the main generator chart at 80 per cent terminal voltage (Fig. 205).

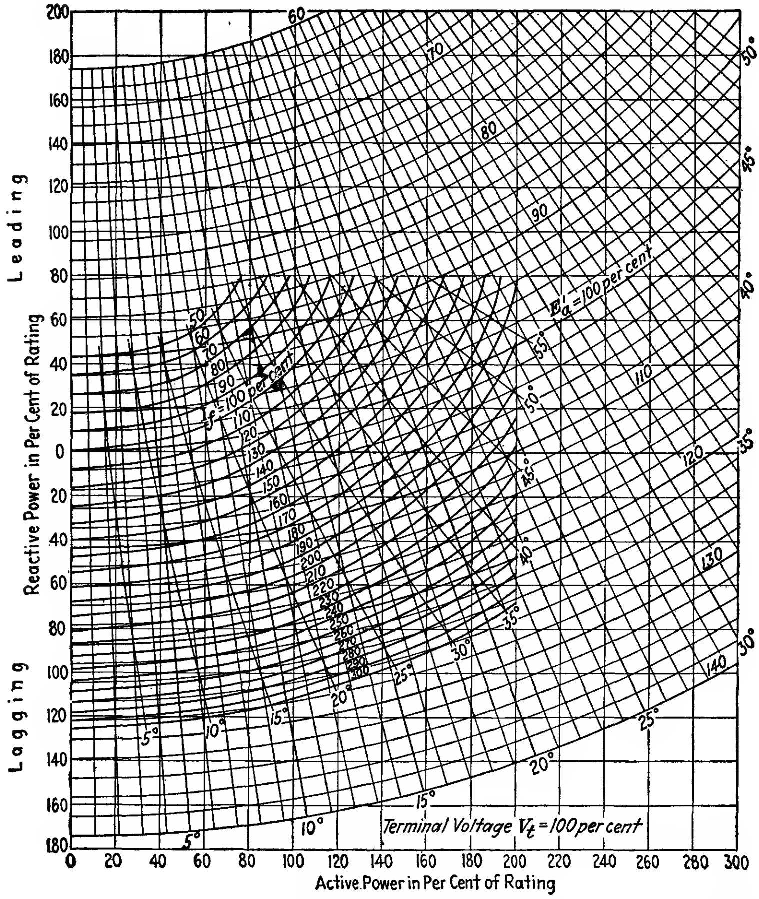

FIG. 201.—Chart for a salient-pole generator at 100 per cen terminal voltage. Circles represent voltage induced by air-gap flux in per cent of normal terminal voltage. Curves represent field current in per cent of field current at normal open-circuit voltage. Straight angle lines represent displacement in electrical degrees between voltage induced by air-gap flux and terminal voltage. Curved angle lines represent displacement in electrical degrees between excitation voltage and terminal voltage. Characteristic curves of this generator are given in Fig. 187 and calculations in Example 1, Tables XVIII and XIX.

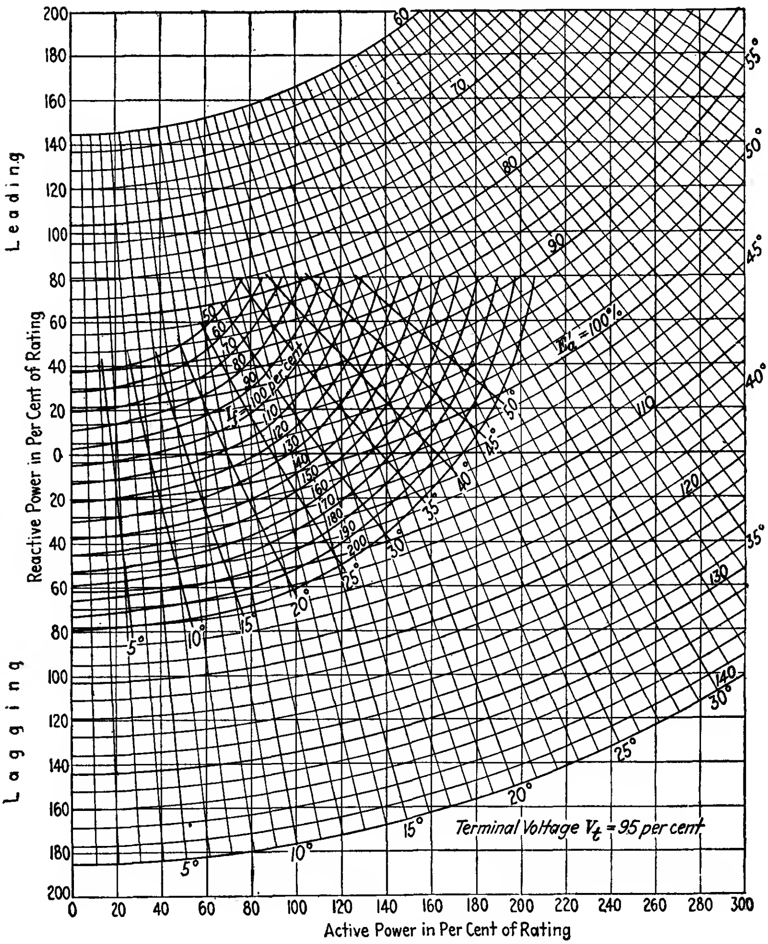

FIG. 202.—Chart for a salient-pole generator at 95 per cent terminal voltage. Circles represent voltage induced by air-gap flux in per cent of normal terminal voltage. Curves represent field current in per cent of field current at normal open-circuit voltage. Straight angle lines represent displacement in electrical degrees between voltage induced by air-gap flux and terminal voltage. Curved angle lines represent displacement in electrical degrees between excitation voltage and terminal voltage. Characteristic curves of this generator are given in Fig. 187.

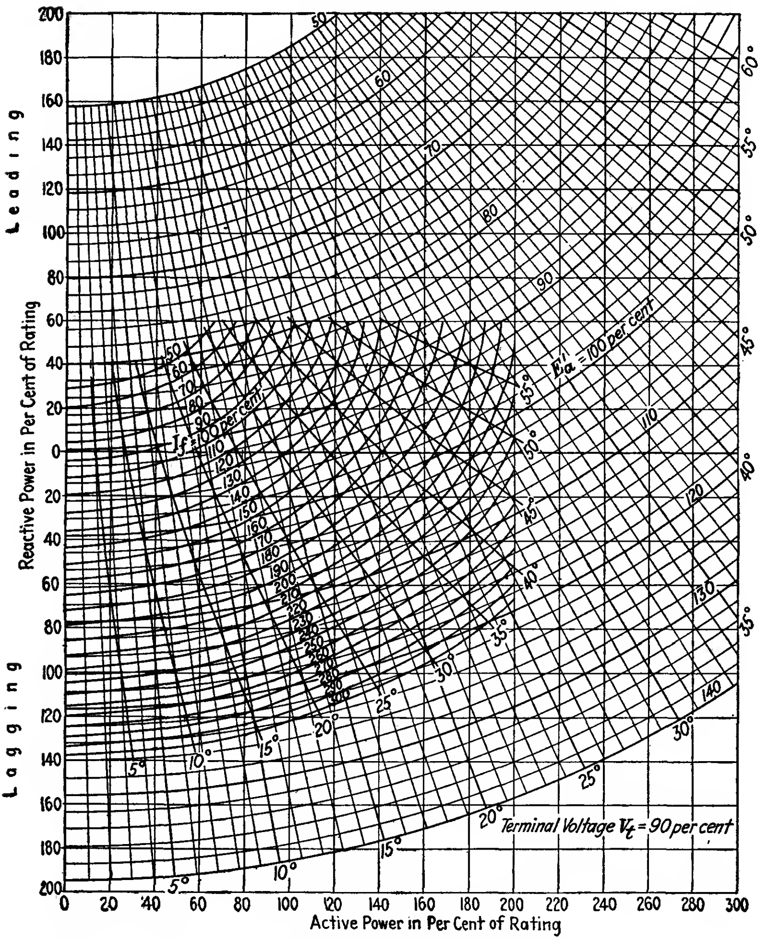

FIG. 203.—Chart for a salient-pole generator at 90 per cent terminal voltage. Circles represent voltage induced by air-gap flux in per cent of normal terminal voltage. Curves represent field current in per cent of field current at normal open-circuit voltage. Straight angle lines represent displacement in electrical degrees between voltage induced by air-gap flux and terminal voltage. Curved angle lines represent displacement in electrical degrees between excitation voltage and terminal voltage. Characteristic curves of this generator are given in Fig. 187, and calculations in Example 1, Tables XVIII and XIX.

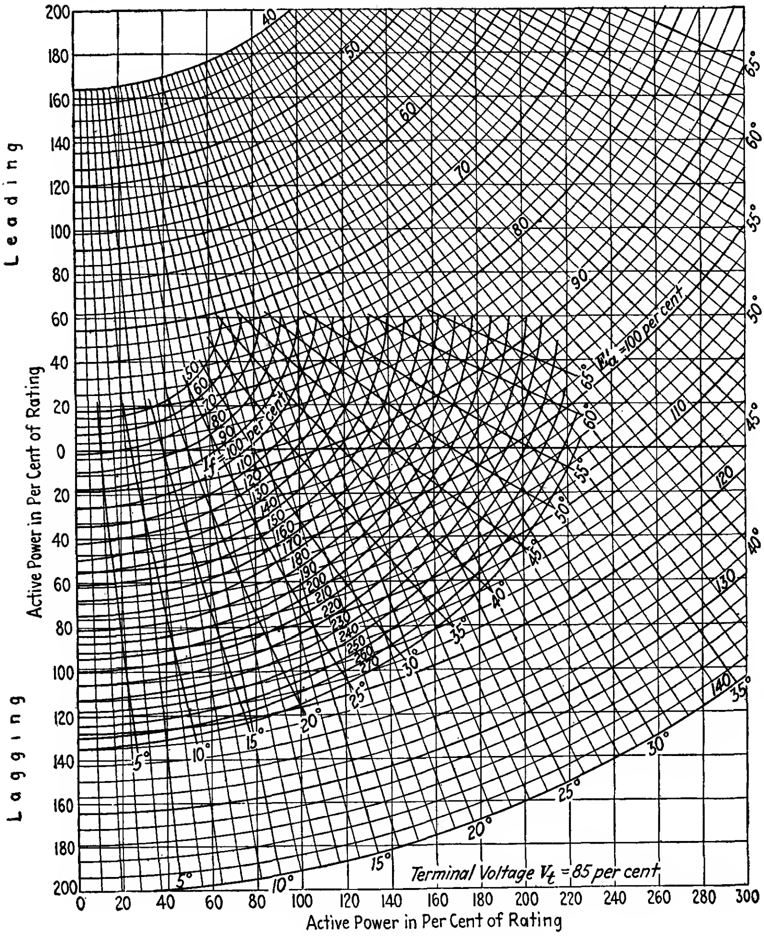

FIG. 204.—Chart for a salient-pole generator at 85 per cent terminal voltage. Circles represent voltage induced by air-gap flux in per cent of normal terminal voltage. Curves represent field current in per cent of field current at normal open-circuit voltage. Straight angle lines represent displacement in electrical degrees between voltage induced by air-gap flux and terminal voltage. Curved angle lines represent displacement in electrical degrees between excitation voltage and terminal voltage. Characteristic curves of this generator are given in Fig. 187.

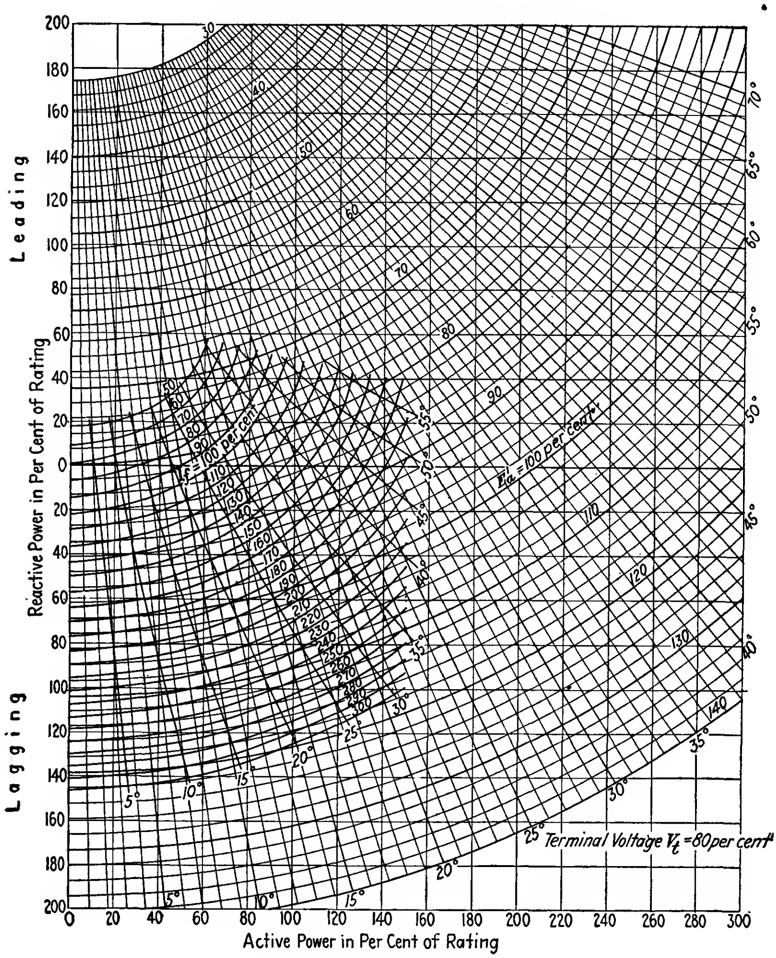

FIG. 205.—Chart for a salient-pole generator at 80 per cent terminal voltage. Circles represent voltage induced by air-gap flux in per cent of normal terminal voltage. Curves represent field current in per cent of field current at normal open-circuit voltage. Straight angle lines represent displacement in electrical degrees between voltage induced by air-gap flux and terminal voltage. Curved angle lines represent displacement in electrical degrees between excitation voltage and terminal voltage. Characteristic curves of this generator are given in Fig. 187, and calculations in Example 1, Tables XVIII and XIX.

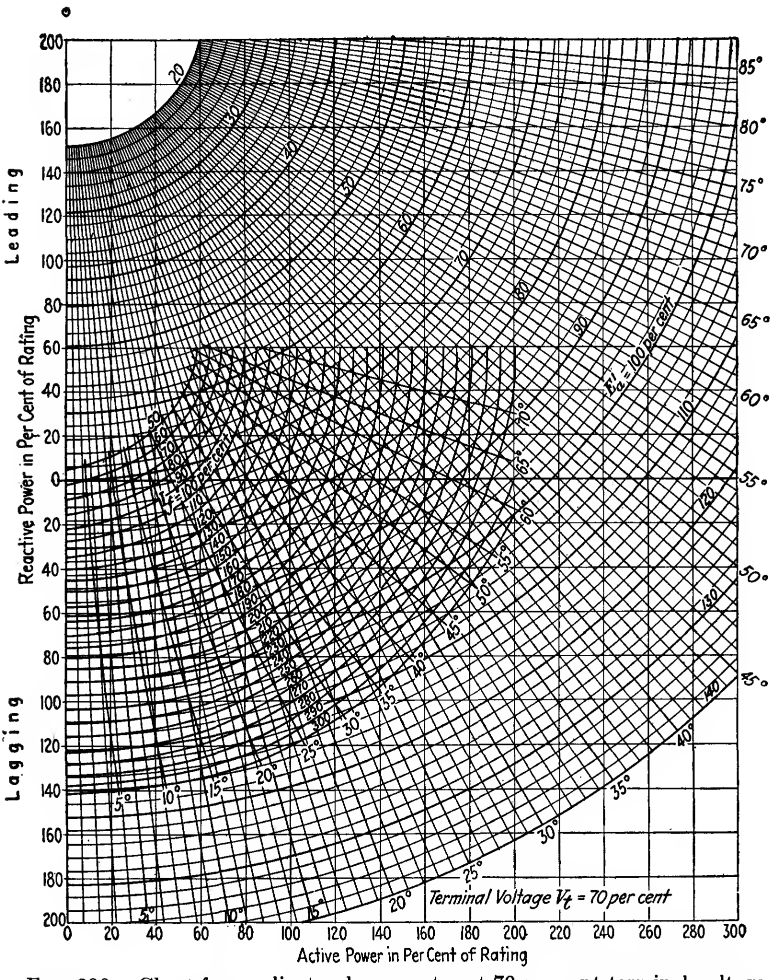

FIG. 206.—Chart for a salient-pole generator at 70 per cent terminal voltage. Circles represent voltage induced by air-gap flux in per cent of normal terminal voltage. Curves represent field current in per cent of field current at normal open-circuit voltage. Straight angle lines represent displacement in electrical degrees between voltage induced by air-gap flux and terminal voltage. Curved angle lines represent displacement in electrical degrees between excitation voltage and terminal voltage. Characteristic curves of this generator are given in Fig. 187.

CHAPTER XII

SOLUTION OF STEADY-STATE TRANSMISSION PROBLEMS

In order to illustrate the solution of problems of electric power transmission in the steady state, particularly by graphical methods, a series of examples is presented below. In most instances, the modified Evans and Sels chart is used to give the performance of the transmission line and transformers. When necessary, the generators are represented by charts of the type described in the preceding chapter.

Determination of Synchronous-condenser Capacity.—In the operation of a transmission line, it is usually required to maintain constant voltage at the receiver end at all loads. This can be done in three ways:

1. By varying the sending voltage.
2. By keeping the sending voltage constant and using a synchronous condenser at the receiving end.
3. By a combination of the two methods just mentioned.

The first method is not practical in long-distance transmission. Constant receiving voltage would, as a rule, necessitate variation in the voltage at the sending end over a very wide range, a feature which is not desirable. At light loads, the sending voltage would have to be quite low in order that the Ferranti effect of the line be properly compensated for. At heavy lagging loads, on the other hand, the sending voltage would have to be very high in order to offset the line drop. Since the factor of safety of the insulators of a high-voltage system cannot be too liberal, for economic reasons, the high voltage might subject the insulators near the sending end of the line to excessive stresses. Furthermore, an undue amount of corona formation might also result on this part of the line. Variation in sending voltage with load can be obtained through control of the generator fields by means of compounded Tirrill regulators.[1]

[1] For description of Tirrill regulators see, for instance, "Handbook for Electrical Engineers," by H. Pender and "Standard Handbook for Electrical Engineers."

In most transmission systems operating today, the voltage is kept constant at the sending end as well as at the receiving end, giving what is usually termed *constant-voltage transmission.*[1] The sending voltage is kept at the desired value (in most cases, equal to the receiving voltage) by means of shunt-wound Tirrill regulators in the field circuits of the generators. The receiving voltage is controlled by synchronous condensers. By field adjustment, as governed by Tirrill regulators, the synchronous condensers may be made to draw a certain amount of leading or lagging current. A leading current taken by the synchronous condensers will produce a drop in the line which counteracts the drop due to a lagging load. If, on the other hand, the synchronous-condenser current is lagging, the drop in the line will tend to compensate for the Ferranti effect and for the drop due to a leading load.

The third method of keeping the receiving voltage constant involves the use of compounded regulators at the sending end and synchronous condensers at the receiving end. The variation of sending voltage is, in this case, limited, so that no undesirable stress is imposed on the insulators. The compounded sending voltage contributes to the maintenance of constant receiving voltage. The additional regulation required at certain loads is taken care of by the synchronous condensers whose fields, as in the previous case, are controlled by shunt-wound Tirrill regulators.[2] It is obvious that the size of the synchronous condenser installation necessary to insure constancy of the receiver voltage is less when the sending voltage is compounded to a certain extent than when this voltage is constant.

A problem which often arises is, therefore, to determine the proper size of the synchronous-condenser installation. If the capacity of the synchronous condensers depended only on their adequacy to give constant receiver voltage at full load and at no load, it could be definitely determined by means of a receiver chart. In deciding, however, on the synchronous-condenser installations in connection with a long transmission line or an

For analysis of operation with compound regulators see L. F. Woodruff, "Regulator Settings for Long Lines," *Elec. World*, Vol. 84, pp. 441 and 460, 1924.

Dwight, H. B., "Transmission Systems with Over-compounded Voltages," *Trans. A.I.E.E.*, p. 95, 1926.

[1] Dwight, H. B, "Constant-voltage Transmission," John Wiley & Sons, Inc., New York, 1915.

[2] See handbooks for electrical engineers, *loc. cit.*

interconnected transmission system, the question of *system stability* must be taken into account. When this factor, which is of extreme importance in large systems, is properly considered, it is entirely possible that synchronous condensers which would have ample capacity for the required steady-state voltage regulation would be wholly inadequate from the standpoint of stability.[1] Hence, only with comparatively short lines and in small systems which inherently possess a considerable stability margin is it safe to base the capacity of the synchronous condensers on considerations of voltage regulation alone.

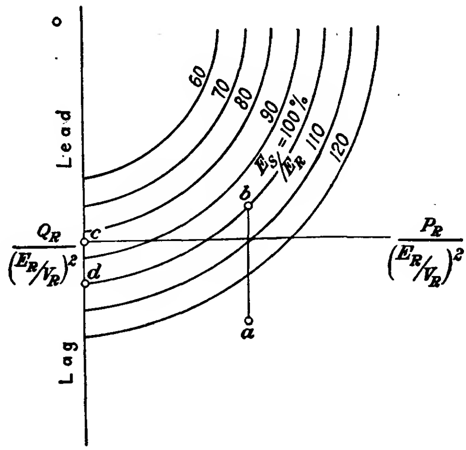

FIG. 207.—Determination of synchronous-condenser capacity by means of the modified Evans and Sels chart.

Evidently, in such cases, the installation should be equal to the capacity required at full load or at no load, depending on which is the larger. As mentioned in the preceding chapter, synchronous condensers sometimes have a slightly smaller rating when operating underexcited than when operating overexcited. If necessary, this must be properly allowed for. Large present-day condensers, however, usually have the same rating over- and underexcited.

Figure 207 shows how the synchronous-condenser capacity can be determined by means of the modified Evans and Sels chart. Assume that the system is to operate with 100 per cent voltage at each end at all loads. Point a is the full-load operating point,

[1] The question of power-system stability is discussed in detail in Vol. II of this treatise.

located by means of active and reactive power of the load. Points b and d are the line operating points at full load and no load, respectively. Neglecting the losses in the synchronous condenser, the vertical line ab gives the required capacity (leading) at full load, and the line cd the required capacity (lagging) at no load. Obviously, the installation should, in this case, be equal to the kilovolt-amperes given by the full-load requirements.

Ignoring the synchronous condenser losses does not introduce any appreciable error.

EXAMPLE 1

Statement of Problem

Line:

Three-phase. Length = 200 miles.

Conductors:

Three aluminum-steel cables spaced 210 in. in a horizontal plane. Diameter 0.954 in. The total number of strands in the concentric-lay cables is 61 (steel – 7; aluminum – 54; all strands same diameter). The conductors are completely transposed.

Load:

Full load = 40,000 kw. at 75 per cent power factor.

Transformers:

Two banks of transformers in parallel at each end of line. Each bank has a capacity of 20,000 kw. at 75 per cent power factor and possesses 0.8 per cent resistance and 12 per cent reactance. The kilovolt-amperes required for excitation are 1,000 kv.-a. per bank, and the core loss 0.5 per cent of the rated capacity (at 75 per cent power factor). The receiving banks step the voltage down to 15 kv.

Transformer Connections:

Δ – Y; neutral grounded.

Conditions of Operation:

Frequency 60 cycles. 150 kv. is held on the high-tension side of the sending transformers by adjustment of the generator fields; 15 kv. is held on the low-tension side of the receiving transformers by adjustment of the synchronous-condenser fields.

Desired Results:

1. Necessary capacity of synchronous condensers to insure constant voltage operation.

2. Plot of reactive kilovolt-amperes of the synchronous condensers *versus* load (at 75 per cent power factor).

Use the chart, which is based directly on the vector diagram. In calculating the resistance of the conductors, neglect the iron core. The skin-effect ratio may, to a sufficient degree of approximation, be considered the same as for a solid aluminum wire of the same diameter. Assume that the iron core increases the internal inductance by 10 per cent. The leakance should be based on the corona loss under average weather conditions.

Solution

Resistance:

The overall diameter (d) of a concentric-lay cable is given by[1]

$$d = d_1(2n + b) \tag{a}$$

where

d_1 = diameter of each strand
n = number of layers over core
b = 1, 2, 2.155, and 2.414 for 1, 2, 3, and 4 wires in the core, respectively.

If the central steel wire, in this case, is considered as the "core," the 61-strand cable has four layers. Hence,

$$d_1 = \frac{0.954}{2 \times 4 + 1} = \frac{0.954}{9} = 0.106 \text{ in.}$$

Area of aluminum

$$A = \frac{54 \times \pi \times 0.106^2}{4} = 0.476 \text{ in.}^2$$

A solid aluminum conductor of 61 per cent conductivity and 0.1662 square in. cross-section (No. 0000 A.W.G.) has a resistance of 0.424 ohm per mile at 20°C.[2] The resistance of the given conductor is, therefore,

$$R' = 0.424 \times \frac{0.1662}{0.476} = 0.1482 \text{ ohm/mile}$$

Correction for Spiraling.—The resistance R' should be corrected for the effect of the spiraling. Obviously, the actual length l of a strand (see Fig. 208) is larger than the length measured axially along the conductor.

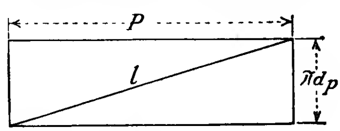

FIG. 208.—Sketch indicating relation between pitch and length of a strand in a concentric-lay cable.

The size of the correction which should be applied to the resistance computed above depends on the average pitch factor of the cable. The pitch factor of any layer is given by

$$p = \frac{l}{P} = \sqrt{1 + \left(\frac{\pi d_p}{P}\right)^2} \tag{b}$$

where

P = the pitch of the layer
d_p = the pitch diameter of the layer
l = the actual length of the strands

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[1] PENDER, H., "Handbook for Electrical Engineers," p. 1873 and following.

[2] See "Handbook for Electrical Engineers," *loc. cit.*

The minimum pitch is 10.1 times the pitch diameter.[1] Hence, for the aluminum part of the given cable,

	d_p, inches	P, inches
Outer layer..............	0.848	8.56
Second layer.............	0.636	6.42
Third layer..............	0.424	4.28

Average pitch factor

$$p = \frac{1}{3}\left[\sqrt{1 + \left(\frac{\pi \times 0.846}{8.56}\right)^2} + \sqrt{1 + \left(\frac{\pi \times 0.636}{6.42}\right)^2} + \sqrt{1 + \left(\frac{\pi \times 0.424}{4.28}\right)^2}\right]$$

$$= 1.048$$

Since, however, there undoubtedly is some conduction axially from strand to strand in each layer, a correction corresponding to the average pitch factor 1.048 would probably be somewhat high. The correction may more appropriately be taken proportional to

$$p' = 1 + k(p - 1) \qquad (c)$$

where k is a factor less than unity, which, in this case, arbitrarily will be assumed equal to $\frac{2}{3}$. Hence,

$$p' = 1 + \frac{2 \times 0.048}{3} = 1.032$$

The corrected d-c. resistance is, therefore,

$$R_0 = p'R' = 1.032 \times 0.1482 = 0.153 \text{ ohm/mile}$$

Skin-effect Ratio.—The "skin-effect" impedance ratio is given by[2]

$$\frac{Z}{R_0} = \frac{\alpha r}{2} \frac{J_0(\alpha r)}{J_1(\alpha r)} \qquad (d)$$

where

$$\alpha = \sqrt{\frac{4\pi\mu\omega}{\rho}} \angle 45° \qquad (e)$$

μ = the permeability of the conductor
ρ = the resistivity of the conductor
$\omega = 2\pi f$ = the angular velocity

The resistivity of aluminum of 61 per cent conductivity is 2.607×10^{-6} ohm-cm. at 0°C. At 20°C.,

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[1] See "Handbook for Electrical Engineers," *loc. cit.*

[2] Kennelly, A. E., F. A. Laws, and H. P. Pierce, "Experimental Researches on Skin Effect in Conductors," *Trans. A.I.E.E.*, p. 1953, 1915.

$$\rho = \rho_0(1 + \alpha_0 t) = 2.607 \times 10^{-6}(1 + 0.00423 \times 20)$$
$$= 2.83 \times 10^{-6} \text{ ohm-cm.} = 2{,}830 \text{ abohm-cm.}$$
$$\alpha = \sqrt{\frac{4\pi \times 377}{2{,}830}}\,\backslash\overline{45°} = 1.279\backslash\overline{45°}$$
$$\alpha r = \frac{1.279 \times 0.954 \times 2.54}{2}\backslash\overline{45°} = 1.55\backslash\overline{45°}$$
$$J_0(\alpha r) = 1.0876\underline{/33°.18}$$
$$J_1(\alpha r) = 0.7868\backslash\overline{27°.86}$$ [1]
$$\frac{Z}{R_0} = \frac{1.55\backslash\overline{45°}}{2} \times \frac{1.0876\underline{/33°.18}}{0.7868\backslash\overline{27°.86}} = 1.071\underline{/16°.0}$$
$$= 1.03 + j0.295$$

Effective resistance of aluminum

$$R = 1.03R_0 = 1.03 \times 0.153$$
$$= 0.1576 = 0.158 \text{ ohm/mile}$$

Internal reactance

$$X_i = 0.295R_0 = 0.295 \times 0.153 = 0.0452 \text{ ohm/mile}$$

Internal inductance

$$L_i = \frac{X_i}{\omega} = \frac{0.0452}{377} = 0.0001198 \text{ henry/mile}$$

This internal inductance is slightly *larger* than the internal inductance obtained on the assumption of uniform current distribution (the latter being 0.0000805 henry/mile). This shows that the formula used is not well applicable to stranded conductors, so far as inductance is concerned. Internal inductance computed from uniform current distribution will, therefore, be used.

Inductance:

Equivalent spacing

$$D = 210\sqrt[3]{2} = 264.8 \text{ in.}$$
$$L = \left(741 \log_{10}\frac{D}{r} + 80.5\right)10^{-6}$$
$$= \left(741 \log_{10}\frac{264.8}{0.477} + 80.5\right)10^{-6}$$
$$= (2{,}033 + 80.5)10^{-6} = 0.002114 \text{ henry/mile}$$

This inductance would be correct if the aluminum conductor had no iron core. As previously stated, it will be assumed that the core increases the internal inductance by 10 per cent. On this basis, the total inductance becomes

$$L = (2{,}033 + 80.5 \times 1.10)10^{-6} = 0.002122$$
$$= 0.00212 \text{ henry/mile}$$

[1] Convenient tables of Bessel functions suitable for skin-effect calculations will be found in the paper "Experimental Researches on Skin Effect in Conductors," by Kennelly, Laws, and Pierce, *loc. cit.* A very complete set of tables is also included in "Funktionentafeln mit Formeln und Kurven," by E. Jahnke and F. Emde, published by B. G. Teubner, Leipzig, 1923. See, also, standard treatises on Bessel functions.

As seen, the difference between the two total inductances is only about one-half of 1 per cent.

Capacitance:

$$C = \frac{0.03882 \times 10^{-6}}{\log_{10} \frac{D}{r}} = \frac{0.03882 \times 10^{-6}}{\log_{10} \frac{264.8}{0.477}} = 0.01412 \times 10^{-6} \text{ farad/mile}$$

Leakance: Leakage over insulators will be assumed negligible.

The corona loss may be computed by assuming

$$\left.\begin{array}{l}\text{Temperature } (t) = 40°\text{F.} \\ \text{Barometric pressure } (b) = 29.7 \text{ in. Hg.}\end{array}\right\} \text{depending upon location}$$

The voltage will, as an approximation, be assumed to be constant equal to 150 kv. over the entire line. Peek's empirical formulas will be used.[1]

For a three-phase line with symmetrical spacing (conductors located at the corners of an equilateral triangle), the corona loss *per mile of conductor* is given by

$$p = \frac{390}{\delta}(f + 25)\sqrt{\frac{r}{D}}\,(e - e_0)^2 10^{-5}\text{kw.} \qquad (f)$$

or by

$$p = \frac{2,070}{\delta}\,(f + 25)\sqrt{\frac{r}{D}}\left(\log_{10}\frac{D}{r}\right)^2 r^2(g - g_0 m_0 \delta)^2 10^{-5}\text{kw.} \qquad (g)$$

Where

$$\delta = \frac{17.9b}{459 + t} = \text{density factor}$$

m_0 = irregularity factor of the wires
f = frequency in cycles per second
r = conductor radius in inches
D = conductor spacing in inches
e = system voltage in kilovolts to neutral
e_0 = disruptive critical voltage in kilovolts to neutral
g = gradient at the conductors in kilovolts per inch
g_0 = 53.6 kv. per inch

The disruptive critical voltage is given by

$$e_0 = g_0 m_0 \delta r \ln\frac{D}{r} = 123 m_0 \delta r \log_{10}\frac{D}{r} \qquad (h)$$

Corona depends fundamentally on the voltage gradient at the conductors. When the spacing is symmetrical, the gradients at the three conductors have identical magnitudes, and each conductor has the same corona loss. In the symmetrical case, therefore, the corona loss may be calculated either by equation (f) in conjunction with equation (h) or by equation (g).

When the spacing is dissymetrical, however, the voltage gradient is not the same at each conductor. The corona loss, therefore, must be calculated separately for each conductor, using equation (g), by computation of the actual voltage gradients at the conductors.[2]

[1] PEEK, F. W., JR., "Dielectric Phenomena in High-voltage Engineering," McGraw-Hill Book Company, Inc., New York, 1920.

[2] See WOODRUFF, L. F., "Principles of Electric Power Transmission and Distribution," *loc. cit.*, Chap. VIII.

In the case of a horizontal arrangement of wires, as in the present case, the maximum gradient occurs at the middle conductor. In order to check up if corona occurs at all, it is sufficient to calculate this gradient. Designating the conductors a, b, and c and letting b represent the middle conductor, the following equations may be written:

$$V_{ab} = V\underline{/0} = r\left[(g_a - g_b)ln\frac{D}{r} - g_c ln 2\right] \qquad (i)$$

$$V_{bc} = V\overline{\backslash 120^\circ} = r\left[(g_b - g_c)ln\frac{D}{r} + g_a ln 2\right] \qquad (j)$$

$$g_a + g_b + g_c = 0 \qquad (k)$$

Simultaneous solution of these equations gives for the voltage gradient at the middle conductor

$$g_b = \frac{V\overline{\backslash 150^\circ}}{\sqrt{3}\, r ln\dfrac{D}{\sqrt[3]{2}\, r}} \qquad (l)$$

Inserting numerical values gives

$$g_b = \frac{150\overline{\backslash 150^\circ}}{\sqrt{3} \times 0.477 \times ln\dfrac{210}{1.26 \times 0.477}} = 31.0\overline{\backslash 150^\circ} \text{ kv. per inch}$$

$$\delta = \frac{17.9 \times 29.7}{459 + 40} = 1.065$$

$$m_0 = 0.85 \text{ (as an average for stranded conductors)}$$

$$g_0 m_0 \delta = 53.6 \times 1.065 \times 0.85 = 48.5 \text{ kv. per inch}$$

Since $g_b < g_0 m_0 \delta$, there is *no corona loss*. The leakance, therefore, will be assumed zero.

Constants to Be Used:

$$\left.\begin{aligned} r &= 0.158 \text{ ohm} \\ g &= 0 \\ L &= 0.00212 \text{ henry} \\ C &= 0.0141 \times 10^{-6} \text{ farad} \end{aligned}\right\} \text{per wire mile}$$

$$z = r + j\omega L = 0.158 + j377 \times 0.00212$$
$$= 0.158 + j0.799 = 0.814\underline{/78^\circ.8} \text{ ohm}$$

$$y = g + j\omega C = j377 \times 0.0141 \times 10^{-6} = j5.315 \times 10^{-6}$$
$$= 5.315 \times 10^{-6}\underline{/90^\circ} \text{ mho}$$

$$\alpha = \sqrt{zy} = 10^{-3}\sqrt{0.814\underline{/78^\circ.8} \times 5.315\underline{/90^\circ}} = 2.08 \times 10^{-3}\underline{/84^\circ.4} \text{ hyp}$$

$$Z_0 = \sqrt{\frac{z}{y}} = 10^3\sqrt{\frac{0.814\underline{/78^\circ.8}}{5.315\underline{/90^\circ}}} = 391\overline{\backslash 5^\circ.6} \text{ ohms}$$

Constants of the Entire Line:

$$Z = 200 \times 0.814\underline{/78^\circ.8} = 162.8\underline{/78^\circ.8} \text{ ohms}$$

$$Y = 200 \times 5.315 \times 10^{-6}\underline{/90^\circ} = 1.063 \times 10^{-3}\underline{/90^\circ} \text{ mho}$$

$$\theta = 200\alpha = 200 \times 2.08 \times 10^{-3}\underline{/84^\circ.4} = 0.416\underline{/84^\circ.4} \text{ hyp}$$

Constants of the Equivalent Π:

$$Z'' = Z\frac{\sinh\theta}{\theta} = 162.8\underline{/78^\circ.8} \times \frac{0.404\underline{/84^\circ.7}}{0.416\underline{/84^\circ.4}} = 158.0\underline{/79^\circ.1}$$

$$= 29.9 + j155.1 \text{ ohms}$$

$$\frac{Y''}{2} = Y\frac{\tanh\frac{\theta}{2}}{\theta} = 1.063 \times 10^{-3}\underline{/90^\circ} \times \frac{0.210\underline{/84^\circ.2}}{0.416\underline{/84^\circ.4}}$$

$$= 0.537 \times 10^{-3}\underline{/89^\circ.2} = (0.00752 + j0.537)10^{-3} \text{ mho}$$

Transformer Constants:

$$I = \frac{20{,}000}{\sqrt{3} \times 150 \times 0.75} = 102.6 \text{ amp. (at full load)}$$

$$\left.\begin{aligned} R_T &= \frac{150{,}000 \times 0.008}{\sqrt{3} \times 102.6} = 6.75 \text{ ohms} \\ X_T &= \frac{150{,}000 \times 0.12}{\sqrt{3} \times 102.6} = 101.2 \text{ ohms} \end{aligned}\right\} \text{per transformer per phase}$$

Since two banks in parallel

$$Z_T = 3.38 + j50.6 \text{ ohms per phase}$$

No-load power factor

$$\cos\theta_n = \frac{20{,}000 \times 0.005}{1{,}000} = 0.100$$

Exciting current

$$I_e = \frac{1{,}000\overline{\backslash\cos^{-1} 0.1}}{\sqrt{3} \times 150} = 3.85(0.1 - j0.995)$$

$$= 0.385 - j3.83 \text{ amp. per bank per phase}$$

Two banks in parallel

$$I_e = 0.77 - j7.66 \text{ amp. per phase}$$

Synchronous Condenser Losses.—Will *assume* size of synchronous condenser = 30,000 kv.-a. and assume that its power factor is 4 per cent. Will further consider the losses in the synchronous condenser to be constant.

On these assumptions, the *constant* current required to supply the losses becomes

$$I_{\text{losses}} = \frac{30{,}000 \times 0.04}{\sqrt{3} \times 150}\underline{/0} = 4.62\underline{/0} \text{ amp. per phase}$$

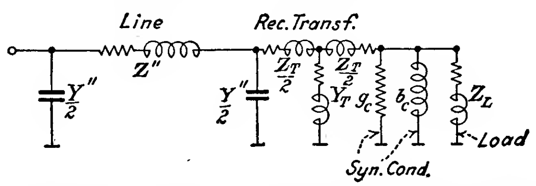


FIG. 209.—Equivalent circuit of system to be solved in Example 1. The circuit includes transmission line, receiving transformers, synchronous condenser, and load.

Solution of the Equivalent Circuit.—The circuit which presents itself for solution is shown in Fig. 209, which, for the sake of simplicity, will be modified to that shown in Fig. 210.

Although the placing of the line leak and the transformer excitation admittance in parallel with the load itself involves an approximation, the accuracy of the solution is not seriously reduced.

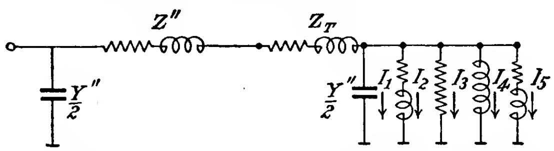


FIG. 210.—Approximate circuit representing system in Fig. 209. As seen, the receiver-end line leak and the transformer excitation admittance have been moved to the point where load and synchronous condenser are attached.

The currents at the receiving end are as follows:

Current I_1 in equivalent line leak.
Current I_2 in transformer excitation admittance.
Current I_3 in synchronous condenser conductance.
Current I_4 in synchronous condenser susceptance.
Current I_5 in load impedance.

Combined equivalent line and transformer impedance

$$Z'' + Z_T = 33.3 + j205.7 \text{ ohms}$$

Currents I_1, I_2, and I_3 are independent of the load and may be combined to a constant current I'.

$$I' = I_1 + I_2 + I_3 = 0.65 + j46.5 + 0.77 - j7.66 + 4.62$$
$$= 6.04 + j38.9 \text{ amp. per phase}$$

Voltage drop in line and transformer impedance due to this current

$$I'(Z'' + Z_T) = (6.04 + j38.9)(33.3 + j205.7)$$
$$= -7{,}800 + j2{,}537 \text{ volts}$$

The receiver diagram, Fig. 211, is based on the calculations above. The necessary synchronous condenser data are obtained from the diagram and collected in the following table:

Load Kilowatts at 75 Per Cent Power Factor	Reactive Kilovolt-amperes of Synchronous Condenser
0	−10,000
5,000	− 4,620
10,000	+ 1,200
15,000	+ 7,180
20,000	+13,480
25,000	+20,240
30,000	+27,100
35,000	+34,400
40,000	+41,700
45,000	+49,800
50,000	+58,300

1. The capacity of the synchronous condenser installation should be at least 41,700 kv.-a.

2. Figure 212 gives the desired relation between reactive kilovolt-amperes of the synchronous condensers and load at 75 per cent power factor.

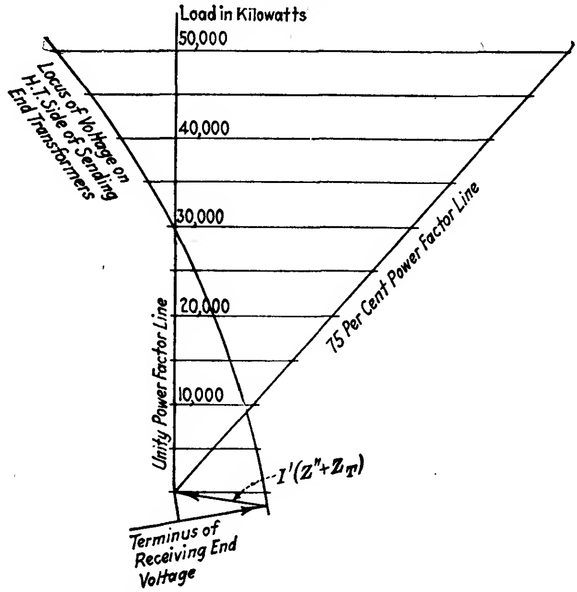


FIG. 211.—Graphical solution of the transmission problem in Example 1. The receiver chart based directly on the vector diagram is used. This type of chart is fully discussed in Chap. X.

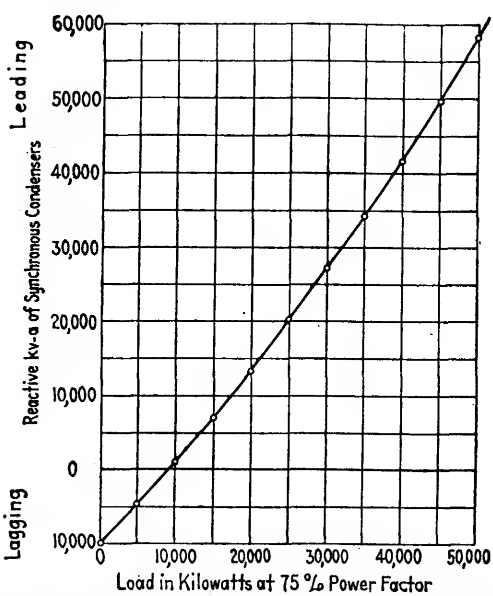


FIG. 212.—Capacity of synchronous condensers necessary to maintain constant receiver voltage for various sizes of load at 75 per cent power factor (lagging). This curve is for the system treated in Example 1.

Determination of Regulation.—The two examples given below illustrate the solution by means of the modified Evans and Sels charts of problems involving voltage regulation.

EXAMPLE 2

Statement of Problem

A 240-mile, 60-cycle transmission line is operated at normal (100 per cent) receiver voltage, the latter being kept constant by adjustment of the sending voltage. The performance of this line including transformers is given by the modified Evans and Sels charts, Figs. 165 and 166.

Calculate the per cent regulation when a load of 100,000 kw. and 45,000 reactive kv.-a. (lagging) is removed.

a. When there is no voltage regulator at the sending end.

b. When the sending voltage is controlled by a compound regulator so adjusted that the sending voltage at no load is 90 per cent.

Solution

a. Since the receiver voltage is normal, *i.e.*, $E_R = V_R = 100$ per cent, the receiver chart is direct-reading and the operating point (100,000 kw. and 45,000 lagging kv.-a.) can be located immediately, and shows that

$$\frac{E_S}{E_R} = 130 \text{ per cent or } E_S = 130 \text{ per cent}$$

When the load is removed

$$\frac{E_S}{E_R} = 85.5 \text{ per cent}$$

Hence,

$$E_R = \frac{130}{0.855} = 152 \text{ per cent}$$

The voltage regulation, therefore, is

$$\text{Regulation} = 152 - 100 = 52 \text{ per cent}$$

b. Upon disconnecting the load, the sending voltage is automatically adjusted to 90 per cent. Hence,

$$E_R = \frac{90}{0.855} = 105.2 \text{ per cent}$$

$$\text{Regulation} = 105.2 - 100 = 5.2 \text{ per cent}$$

In this case, the regulation is obviously independent of the amount of load which is thrown off.

EXAMPLE 3

Statement of Problem

A three-phase, 240-mile, 60-cycle, 220-kv. transmission line is normally operated at terminal voltages of 220 kv. The low-tension voltage at the sending end is maintained by automatic field control of the generators while the voltage of the low-tension bus at the receiving end is regulated by three 30,000-kv.-a. synchronous condensers. The performance of the line including transformers is given by the modified Evans and Sels charts (Figs. 165 and 166). The characteristics of the synchronous condensers are given in Fig. 186.

a. What load at 90 per cent power factor (lagging) can the line supply when the synchronous condensers operate at 85 per cent field current and the low-tension sending and receiving voltage are both the equivalent of 220 kv.?

b. Assuming the Tirrill regulators adjusting the fields of the synchronous condensers to be out of service (so that the field current remains constant), calculate the voltage regulation of the line when the 90 per cent power factor load determined in (*a*) is thrown off.

Solution

a. From the V-curves of the synchronous condensers, it is seen that at 100 per cent voltage and 85 per cent field current, the leading kilovolt-amperes taken are 68.6 per cent or 61,700 kv.-a. (100 per cent is 90,000 kv.-a.).

If a 90 per cent power-factor line is drawn in the receiver chart, the load operating point is located where the vertical distance between the power-factor line and the 100 per cent voltage circle equals 61,700 kv.-a. This gives

$$P_R = 95{,}500 \text{ kw. at 90 per cent power factor}$$

b. When the load is removed and the field currents of the synchronous condensers are kept constant, the receiving voltage will change until the reactive kilovolt-amperes supplied by the line equal the reactive kilovolt-amperes taken by the synchronous condensers at that particular voltage. In order to locate definitely the new operating point, it is best to prepare an auxiliary plot of line kilovolt-amperes and condenser kilovolt-amperes *versus* receiving voltage and obtain the intersection between the two curves.

Tables XX and XXI give synchronous-condenser data as obtained from the V-curves at 85 per cent field current, and line data at no load, as obtained from the receiver chart.

TABLE XX.—SYNCHRONOUS-CONDENSER DATA (EXAMPLE 3)

V_t, per cent	I_a, per cent	Q_c, per cent	Q_c, kilovolt-amperes
90	85.0	76.5	68,800
100	68.6	68.6	61,700
110	50.0	55.0	49,500
120	27.3	32.8	29,500
130	4.2	5.46	4,910
140	−28.5	−39.9	−35,900

TABLE XXI.—LINE DATA AT NO LOAD (EXAMPLE 3)

E_S, per cent	$\frac{E_S}{E_R}$, per cent	E_R, per cent	$\frac{Q_R}{(E_R/V_R)^2}$, kilovolt-amperes	Q_R, kilovolt-amperes
100	70	143	26,200	53,500
100	80	125	9,500	14,850
100	90	111.1	− 8,000	− 9,880
100	100	100	−26,000	−26,000
100	110	91	−43,300	−35,800

Figure 213 shows the plotted curves. As seen, they intersect at 125.9 per cent of normal receiving voltage. Hence,

Regulation = 125.9 − 100 = 25.9 per cent

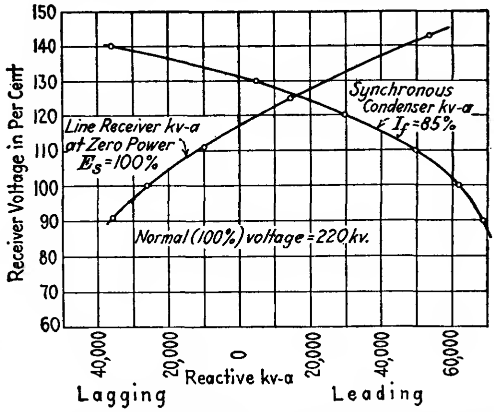


FIG. 213.—Determination of voltage regulation by obtaining intersection between locus of reactive power at the receiver end of the line and locus of reactive power of the synchronous condensers at constant field current. These curves give the solution of Example 3, Part (*b*).

Determination of Load Division between Two Parallel Lines.—A load is very often supplied over two parallel lines from the same generating station or substation. If the parallel lines have identical electrical constants, they will obviously divide the load equally. If, on the other hand, the lines have widely different constants, they will not carry an equal share of the load. In such cases, the problem of determining the load division between the two lines arises.

The load division can be determined by calculation or by the use of charts. The example below discusses some methods which may be used to determine the division of load between the parallel lines in a specific case.

EXAMPLE 4

Statement of Problem

A generating station and a substation, a considerable distance apart, are connected by two three-phase transmission lines on separate pole lines on separate rights of way. The constants of the two lines are widely different. Both lines are tied into the high-tension buses at each end. The voltages on the high-tension side of the sending transformers and on the low-tension side of the receiving transformers have their normal (100 per cent) values.

Knowing the constants of the lines and transformers, the power and power factor (lagging) of the load, outline the procedure to be followed in order to determine the load division between the two lines.

Solution

1. By calculation using general circuit constants.

General circuit constants of line 1: A_1, B_1, C_1, D_1.

General circuit constants of line 2: A_2, B_2, C_2, D_2.

General circuit constants of the two lines in parallel [Chap. IX, equations (58)]

$$A_0' = \frac{A_1B_2 + B_1A_2}{B_1 + B_2} \tag{a}$$

$$B_0' = \frac{B_1B_2}{B_1 + B_2} \tag{b}$$

$$C_0' = C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \tag{c}$$

$$D_0' = \frac{B_1D_2 + D_1B_2}{B_1 + B_2} \tag{d}$$

Letting Z_r and Y_r represent the equivalent impedance and excitation admittance, respectively, of the receiving transformers, the general circuit constants of the two lines in series with the receiving transformers, *i.e.*,

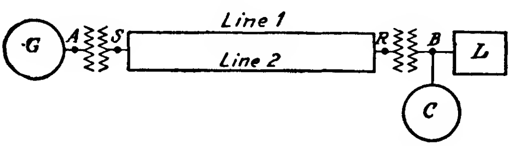


FIG. 214.—System considered in Example 4.

of the system between points S and B (see Fig. 214), become (Chap. IX, equations (75))

$$A_0 = A_0'\left(1 + \frac{Z_rY_r}{2}\right) + B_0'Y_r \tag{e}$$

$$B_0 = B_0'\left(1 + \frac{Z_rY_r}{2}\right) + A_0'Z_r\left(1 + \frac{Z_rY_r}{4}\right) \tag{f}$$

$$C_0 = C_0'\left(1 + \frac{Z_rY_r}{4}\right) + D_0'Y_r \tag{g}$$

$$D_0 = D_0'\left(1 + \frac{Z_rY_r}{2}\right) + C_0'Z_r\left(1 + \frac{Z_rY_r}{4}\right) \tag{h}$$

The sending voltage of the lines (at S) is given. In order to obtain the distribution of power between them, the receiving voltage (at R) must be calculated. First it is necessary, however, to determine the current taken by the synchronous condenser in order that the total current at B may be obtained. Since the power factor of the load is lagging, it will be assumed that the synchronous condenser current (I_C) is leading.

$$\begin{aligned} E_s &= A_0E_b + B_0I_b = A_0E_b + B_0(I_L + I_C) \\ &= (a_1 + ja_2)E_b + (b_1 + jb_2)(I_{Lp} - jI_{Lq} + jI_C) \\ &= (a_1E_b + b_1I_{Lp} + b_2I_{Lq} - b_2I_C) \\ &\quad + j(a_2E_b + b_2I_{Lp} - b_1I_{Lq} + b_1I_C) \\ &= (M - b_2I_C) + j(N + b_1I_C) \end{aligned} \tag{i}$$

Since the absolute value of E_s is known, this equation is readily solved by squaring and gives for the synchronous condenser current

$$I_C = -\frac{Nb_1 - Mb_2}{b_1^2 + b_2^2} \pm \sqrt{\left[\frac{Nb_1 - Mb_2}{b_1^2 + b_2^2}\right]^2 - \frac{M^2 + N^2 - E_s^2}{b_1^2 + b_2^2}} \qquad (j)$$

The sum of this current and the load current is the total current at B. The voltage and current on the high-tension side of the receiving transformers are given by

$$E_r = \left(1 + \frac{Z_rY_r}{2}\right)E_b + Z_r\left(1 + \frac{Z_rY_r}{4}\right)I_b \qquad (k)$$

$$I_r = Y_rE_b + \left(1 + \frac{Z_rY_r}{2}\right)I_b \qquad (l)$$

So far, the load voltage E_b has been taken as reference vector. In the following, it is convenient to use E_r for this purpose, and I_r, as found from equation (l), should, therefore, be inserted in the equation below in its proper phase relationship with respect to this voltage.

$$E_s = A_1E_r + B_1I_{r1} = A_2E_r + B_2I_{r2} \qquad (m)$$

which, in conjunction with

$$I_{r1} + I_{r2} = I_r \qquad (n)$$

gives

$$I_{r1} = \frac{(A_2 - A_1)E_r + B_2I_r}{B_1 + B_2} \qquad (o)$$

$$I_{r2} = \frac{(A_1 - A_2)E_r + B_1I_r}{B_1 + B_2} \qquad (p)$$

Since I_{r1} and I_{r2} are obtained with reference to E_r, the phase displacements ϕ_{r1} and ϕ_{r2} at the receiving ends of the two lines are readily determined. The total amounts of power supplied to the substation transformers by the lines are, therefore,

$$P_{R1} = 3E_rI_{r1}\cos\phi_{r1} \qquad (q)$$

$$P_{R2} = 3E_rI_{r2}\cos\phi_{r2} \qquad (r)$$

2. By calculations based on equivalent Π*-circuits.*

As shown in Fig. 215, each line has been replaced by its equivalent Π. In order to determine the synchronous-condenser current, the two Π-circuits

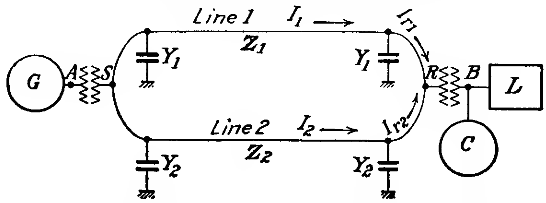


FIG. 215.—Equivalent circuit of the system in Fig. 214. The two parallel transmission lines have been replaced by their equivalent Π-circuits.

are combined into a single Π, as indicated in Fig. 216, and the receiving transformers replaced by their equivalent T-circuit.

The voltage and current on the high-tension side of the receiving transformers are given by equations (k) and (l) as before. The voltage on the high-tension side of the sending transformers is

$$E_s = \left(1 + \frac{Z_0 Y_0}{2}\right)E_r + Z_0 I_r \tag{s}$$

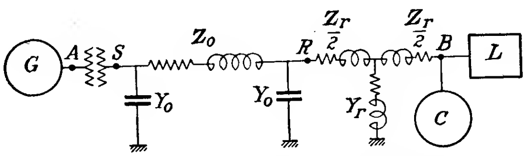


FIG. 216.—Circuit representing the system in Fig. 215 obtained by combining the two equivalent Π's in parallel and representing the receiver transformers by their equivalent T-circuit.

which, when combined with equations (k) and (l), changes into

$$E_s = \left[\left(1 + \frac{Z_0Y_0}{2}\right)\left(1 + \frac{Z_rY_r}{2}\right) + Z_0Y_r\right]E_b + \left[Z_r\left(1 + \frac{Z_0Y_0}{2}\right)\left(1 + \frac{Z_rY_r}{4}\right) + Z_0\left(1 + \frac{Z_rY_r}{2}\right)\right]I_b \tag{t}$$

This equation, when reduced, may be written

$$\begin{aligned} E_s &= (a_1 + ja_2)E_b + (b_1 + jb_2)I_b \\ &= (a_1 + ja_2)E_b + (b_1 + jb_2)(I_{bp} - jI_{bq} + jI_C) \end{aligned} \tag{u}$$

Comparison shows that equations (u) and (i) are identical. The synchronous-condenser current, therefore, is determined by equation (j).

Knowing the total current ($I_L + I_C$) on the load side of the receiving transformers, the voltage and current on their high-tension side are calculated by equations (k) and (l). The ratio of the currents carried by the architraves of the two equivalent Π's (Fig. 215) equals the inverse ratio of the architrave impedances. Thus,

$$\frac{I_1}{I_2} = \frac{Z_2}{Z_1} \tag{v}$$

The sum of the architrave currents is given by

$$I_1 + I_2 = (Y_1 + Y_2)E_r + I_r \tag{w}$$

Solving for I_1 and I_2 from equations (v) and (w) gives

$$I_1 = \frac{Z_2(Y_1 + Y_2)E_r + Z_2I_r}{Z_1 + Z_2} \tag{x}$$

$$I_2 = \frac{Z_1(Y_1 + Y_2)E_r + Z_1I_r}{Z_1 + Z_2} \tag{y}$$

The current at the receiver end of each line is, hence,

$$I_{r1} = \left[\frac{Z_2(Y_1 + Y_2)}{Z_1 + Z_2} + Y_1\right]E_r + \frac{Z_2}{Z_1 + Z_2}I_r \tag{z}$$

$$I_{r2} = \left[\frac{Z_1(Y_1 + Y_2)}{Z_1 + Z_2} + Y_2\right]E_r + \frac{Z_1}{Z_1 + Z_2}I_r \tag{aa}$$

and the power supplied to the substation transformers by the lines is given by equations (q) and (r).

3. By the use of charts.

In order to solve this problem by means of charts, it is necessary to have at least three charts available, namely, a receiver chart for the system between the high-tension bus of the sending transformers and the low-tension

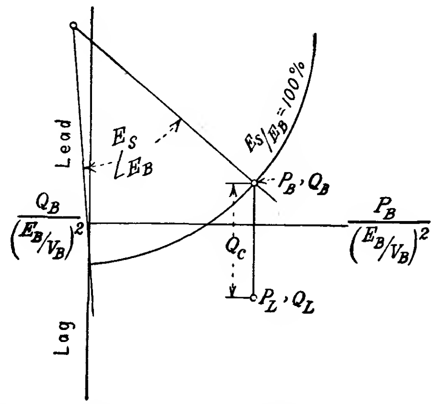
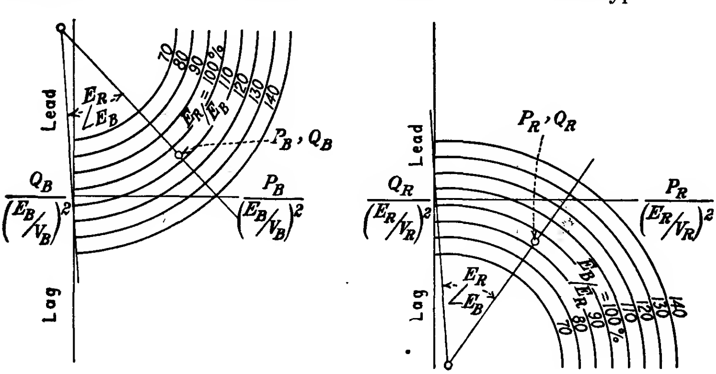


FIG. 217.—Modified Evans and Sels receiver chart for the system in Fig. 214 between the high-tension bus at the sending end (S) and the low-tension bus at the receiving end (B).

bus of the receiving transformers (*i.e.*, between points S and B) and one receiver chart for each transmission line proper (*i.e.*, between S and R). Assume that these charts are of the modified Evans and Sels type.



(a) (b)

FIG. 218.—Modified Evans and Sels charts for the performance of the receiving-end transformers.

By spotting active and reactive power taken by the load in the chart for the system between S and B (Fig. 217), the synchronous-condenser kilovolt-amperes and the active and reactive power (P_B and Q_B) at the low-tension side of the receiving transformers are determined. The current at this point

may now be calculated, and the voltage and current on the high-tension side of the transformer obtained by means of equations (*k*) and (*l*). Having the voltage and current, the active and reactive power are also readily calculated. This power and kilovolt-amperes, however, can be determined

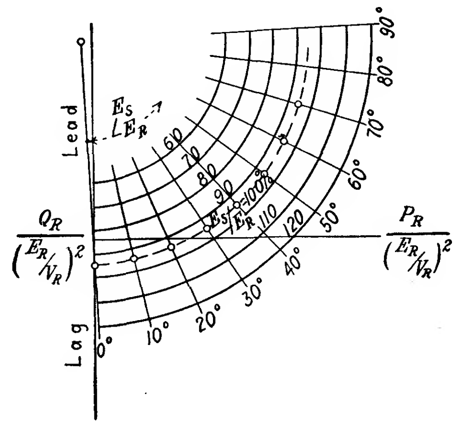


FIG. 219.—Modified Evans and Sels receiver chart for one of the transmission lines between high-tension buses. Operating points must lie on the dotted circle representing the correct ratio between sending and receiving voltage.

still more readily if performance charts for the receiving transformers alone (*i.e.*, between points *R* and *B*) are available.

Figure 218 shows a receiving and sending chart for these transformers. The operating point (P_B and Q_B) can be directly spotted in the receiving chart *a* and transferred to the sending chart *b* by means of voltage ratio and

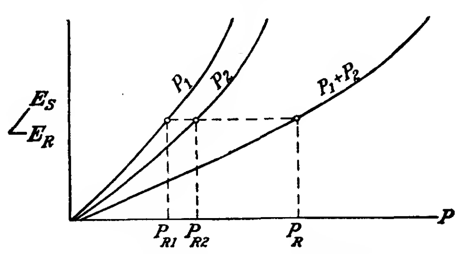


FIG. 220.—Graphical determination of load distribution between the two parallel lines.

angle. This gives the voltage and active and reactive power (E_R, P_R, and Q_R) on the high-tension side of the receiving transformers.

Figure 219 shows a receiving chart for one of the transmission lines. Since E_R is known, the ratio E_S/E_R is given and, hence, the circle on which the operating point must lie. This circle is shown dotted. The load will divide between the two lines in such a manner that the following two conditions are satisfied:

1. The sum of the power supplied by each line must equal the total power at R.

2. The displacement between the sending and receiving voltages must be the same for the two lines.

When the power is determined in accordance with these conditions, the values obtained for the reactive power will be such that the condition that the reactive power supplied by the two lines must equal the total reactive power at R will automatically be satisfied.

The correct power division can be obtained most conveniently by an auxiliary plot (Fig. 220). Values of power corresponding to points along the proper voltage-ratio circle in the two receiving charts are plotted *versus* angular displacement. By addition, a curve of the total power *versus* angle is obtained. On this curve, the correct P_R is spotted, and the amounts of power P_{R1} and P_{R2} supplied by each line determined, as indicated in the figure.

Miscellaneous Examples.—In order to illustrate still further the use of charts for the solution of steady-state transmission problems, a few examples are given below. These involve the determination of voltages, loads, power, and power factor of generators, reactive kilovolt-amperes taken by synchronous condensers, field currents of generators, and synchronous condensers, etc.

EXAMPLE 5

Statement of Problem

A generating station G supplies power to two loads L_1 and L_2 over the long transmission lines AM and MB (Fig. 221). The sections AM and MB

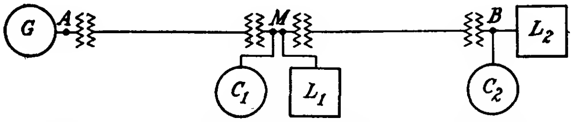


FIG. 221.—Transmission system considered in Example 5.

are of equal length and identical in every respect. Their steady-state performance is given by the modified Evans and Sels charts (Figs. 165 and 166).

The voltage at A is regulated by automatic field control of the generators and is normally 110 per cent. The voltages at M and B are regulated by the synchronous condensers C_1 and C_2 and are normally 100 per cent. The nominal (100 per cent) voltage is 220 kv. referred to the high-tension sides. The load L_1 is 100,000 kw. at 85 per cent power factor (lagging); and the load L_2, 40,000 kw. at unity power factor.

a. For operation as specified above, what are the reactive kilovolt-amperes taken by the synchronous condensers? What are the power and the power factor at the generators?

b. What would be the effect on the steady-state voltage, power, and power factor at A if the synchronous condenser C_1 is shut down? (The loads L_1 and L_2 are as before, and the voltages at M and B are 100 per cent of normal.)

Solution

a. Entering the receiver chart with 40,000 kw. at unity power factor, the reactive kilovolt-amperes taken by the synchronous condenser C_2 is obtained as the vertical distance between the load operating point and the 100 per cent voltage circle. The angle between the sending and receiving voltage is also read. This gives

$$\text{Line } MB \begin{cases} P_R = P_{L2} = 40{,}000 \text{ kw.} \\ Q_R = Q_{C2} = -16{,}300 \text{ kv.-a.} \\ \angle\dfrac{E_S}{E_R} = 13.2 \text{ deg.} \end{cases}$$

By entering the sending chart with the angular displacement 13.2 deg., the sending operating point of the line MB is found on the 100 per cent voltage circle. Thus,

$$\text{Line } MB \begin{cases} P_S = 44{,}000 \text{ kw.} \\ Q_S = +25{,}000 \text{ kv.-a.} \end{cases}$$

The load at M takes 100,000 kw. at 85 per cent power factor. The reactive power taken by this load is consequently $-62{,}000$ kv.-a. The power received over the line AM is 144,000 kw. Entering the receiver chart with 144,000 kw., the reactive kilovolt-amperes supplied by this line are found to be $+45{,}000$ kv.-a. It is given by the vertical distance between the horizontal axis and the 110 per cent voltage circle. The angular displacement is also read.

Hence,

$$\text{Line } AM \begin{cases} P_R = 144{,}000 \text{ kw.} \\ Q_R = +45{,}000 \text{ kv.-a.} \\ \angle\dfrac{E_S}{E_R} = 50.3 \text{ deg.} \end{cases}$$

The reactive kilovolt-amperes of the synchronous condenser C_1 are now given by

$$\begin{aligned} Q_{C1} &= Q_R - Q_S - Q_L \\ &= 45{,}000 - 25{,}000 + 62{,}000 = +82{,}000 \text{ kv.-a.} \end{aligned}$$

The sending operating point of the line AM is located in the sending chart by means of angular displacement 50.3 deg. and voltage ratio which now is

$$\frac{E_R}{E_S} = \frac{100}{1.10} = 90.9 \text{ per cent}$$

It should be remembered that the sending voltage in this case is different from the nominal voltage; the chart readings must be multiplied by the square of the ratio of these voltages. Hence,

$$\begin{aligned} P_S &= 132{,}000 \times 1.10^2 = 160{,}000 \text{ kw.} \\ Q_S &= -37{,}000 \times 1.10^2 = -44{,}800 \text{ kv.-a.} \end{aligned}$$

$$\cos\phi_S = \frac{100}{\sqrt{1 + \left(\dfrac{44.8}{160}\right)^2}} = 96.3 \text{ per cent (lagging)}$$

The desired results are

$$\begin{aligned} Q_{C2} &= -16{,}300 \text{ kv.-a.} \\ Q_{C1} &= +82{,}000 \text{ kv.-a.} \\ P_A &= 160{,}000 \text{ kw.} \\ \text{Power factor at } A &= 96.3 \text{ per cent (lagging)} \end{aligned}$$

b. When the synchronous condenser C_1 is shut down, the power received over the line AM remains 144,000 kw. The reactive kilovolt-amperes, however, are now $-37{,}000$ kv.-a., namely, the sum of the reactive kilovolt-amperes taken by the load L_1 and the reactive kilovolt-amperes supplied to the line MB. Locating the receiver operating point as given by the above given values of power and reactive kilovolt-amperes in the receiver chart, the ratio of the voltages at A and M and the angle between them are read. Hence,

$$\text{Line } AM \begin{cases} P_R = 144{,}000 \text{ kw.} \\ Q_R = -37{,}000 \text{ kv.-a.} \\ \dfrac{E_S}{E_R} = 141.5 \text{ per cent} \\ \angle\dfrac{E_S}{E_R} = 34.5 \text{ deg.} \end{cases}$$

The sending chart is now entered by the angular displacement 34.5 deg. and the ratio

$$\frac{E_R}{E_S} = \frac{100}{1.415} = 70.6 \text{ per cent}$$

This gives

$$\begin{aligned} P_S &= 79{,}000 \times 1.415^2 = 158{,}000 \text{ kw.} \\ Q_S &= -40{,}500 \times 1.415^2 = -81{,}000 \text{ kv.-a.} \\ \cos\phi_S &= \frac{100}{\sqrt{1 + \left(\frac{81}{158}\right)^2}} = 89.0 \text{ per cent (lagging)} \end{aligned}$$

The desired results are

$$\begin{aligned} E_A &= 141.5 \text{ per cent} \\ P_A &= 158{,}000 \text{ kw.} \\ \text{Power factor at } A &= 89.0 \text{ per cent (lagging)} \end{aligned}$$

Hence, the shutting down of the synchronous condenser affects the power at A but very little. The voltage and power factor, however, are affected to a considerable extent. The voltage has to be materially increased to transmit the desired power, while the power factor is somewhat decreased.

EXAMPLE 6

Statement of Problem

A hydro generating station of known kilovolt-ampere capacity at A supplies power over the transmission line AB to the load L_B and over the line MN to the load L_N (Fig. 222). A steam generating station is feeding into the system at B. Synchronous condensers C_B and C_N of ample capacity are installed at the loads. The voltages at the points A, $B(M)$, and N have their normal (100 per cent) values.

The power taken by the load L_B and its power factor (lagging) are known. Assuming that modified Evans and Sels charts for the two transmission lines, charts at 100 per cent terminal voltage for the generators, and V-curves for the synchronous condensers are available, describe the procedure which should be followed in order to determine

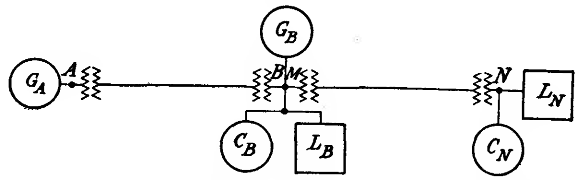


FIG. 222.—Transmission system considered in Example 6.

a. The maximum unity power-factor load L_N which can be carried in the steady state at N without overloading the generators.

b. The power and reactive kilovolt-amperes and the field current of the hydro generators.

c. The reactive kilovolt-amperes and field current of the synchronous condensers.

Solution

a. The load on any generator is limited by its kilovolt-ampere rating. It will supply maximum power, therefore, when operating at unity power factor, provided such operation is possible.

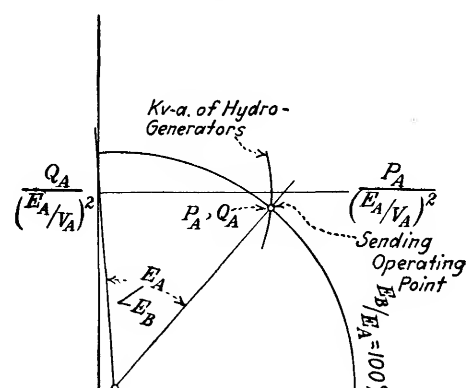


FIG. 223.—Determination of active and reactive power at point A in the system in Fig. 222. The operating point is obtained at the intersection between the proper voltage-ratio circle in a modified Evans and Sels sending chart and an arc representing rated kilovolt-amperes of the hydrogenerators.

The hydro generators may not be able to supply power at unity power factor, since fixing the voltages at A and B also fixes the sending reactive kilovolt-amperes of the line AB corresponding to any particular value of power. Obviously, the sending reactive kilovolt-amperes of the line and the

reactive kilovolt-amperes of the generators must be the same. The sending operating point is located by striking an arc about the origin of the sending chart with a radius equal to the kilovolt-ampere capacity of the hydro generators, as shown in Fig. 223.

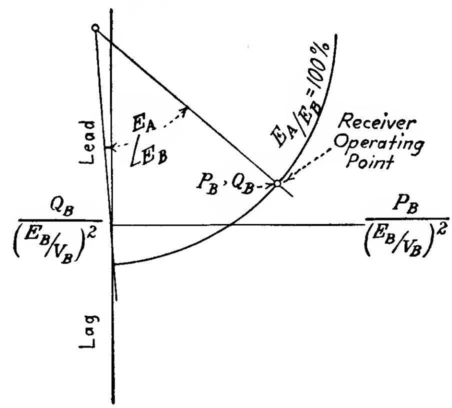


FIG. 224.—Modified Evans and Sels receiver chart for the line A-B (see Fig. 222). From this chart the active and reactive power at B are obtained.

The sending operating point of the line AB is transferred to the receiver chart for this line Fig. 224 by means of voltage ratio (E_A/E_B = 100 per cent) and angular displacement between E_A and E_B.

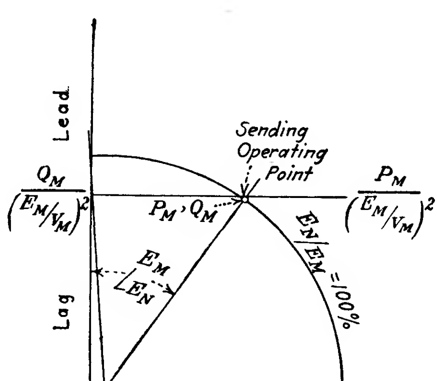


FIG. 225.—Modified Evans and Sels sending chart for the line M-N (see Fig. 222). The active and reactive power at M are located in this chart and the displacement between the terminal voltages read so as to make transfer to the receiver chart possible.

Since there is a synchronous condenser at B, the steam generators can be operated at unity power factor and, hence, supply to the system an amount of power equal to their kilovolt-ampere rating.

The power supplied to the line MN is given by

$$P_M = P_B + P_{GB} - P_{LB} \qquad (a)$$

The sending operating point of the line MN is located as shown in Fig. 225 by means of power P_M and voltage ratio (E_N/E_M = 100 per cent) and then transferred to the receiver chart in the ordinary manner (Fig. 226).

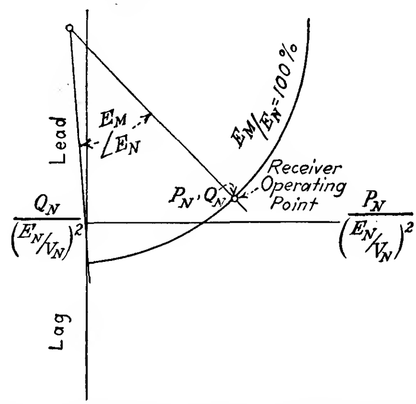


FIG. 226.—Modified Evans and Sels receiver chart for the line M-N (see Fig. 222). From this chart the active and reactive power at N are obtained.

The value of the power P_N so determined is the maximum unity-power-factor load which can be connected at N without overloading the generators. Hence,

$$L_N = P_N \qquad (b)$$

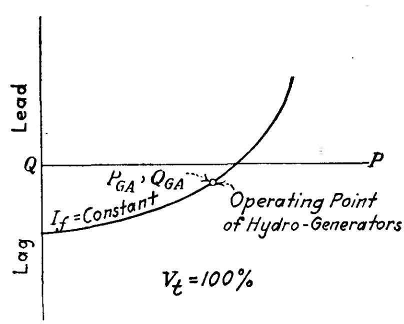


FIG. 227.—Diagram showing how the field current of the hydrogenerators may be determined by locating the active and reactive power at A (see Fig. 222) in the generator chart for the proper terminal voltage (solution of Example 6, Part (b)).

It should be noted that the maximum amount of power which can be supplied to the load at N is actually *independent of the power factor* of this load, provided, of course, the synchronous condensers at this point are sufficiently large to compensate for the reactive kilovolt-amperes.

b. The power and reactive kilovolt-amperes of the hydro generators are found at the sending operating point of the line AB (Fig. 223). By locating a point corresponding to this power and reactive power in the generator chart for 100 per cent terminal voltage (Fig. 227), the field current of the hydro generators is immediately determined.

c. The reactive kilovolt-amperes taken by the synchronous condensers at B and N are found by

$$Q_{CB} = Q_B - Q_{LB} - Q_M \tag{c}$$

and

$$Q_{CN} = Q_N \tag{d}$$

where Q_M is the sending reactive power of the line MN as found in Fig. 225, Q_B and Q_N are the receiving reactive power of lines AB and MN, respec-

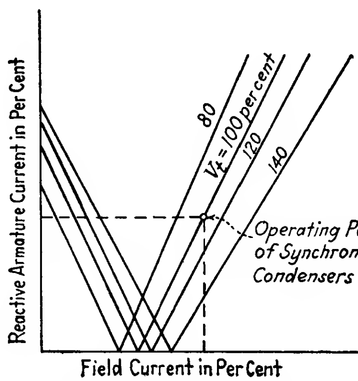


FIG. 228.—Diagram showing how the field current of the synchronous condensers are determined by means of a family of V-curves (solution of Example 6, Part (*c*)).

tively, as determined in Figs. 224 and 226, and Q_{LB} is the reactive power taken by the load at B.

Having determined the reactive kilovolt-amperes of the synchronous condensers, their field currents are determined from the V-curves. Assuming that the kilovolt-amperes taken are leading, the condenser operating point is located on the 100 per cent voltage curve, as shown in Fig. 228, and the field current can immediately be read

EXAMPLE 7

Statement of Problem

Work out the numerical solution of Example 6, using the following data:

Hydro Generating Station:

Installation: 4 − 32,000 kv.-a. generators. Performance given by the chart (Fig. 201).

Steam Generating Station:

Installation: 2 − 30,000 kv.-a. generators.

Transmission Lines and Transformers:

Performance given by the modified Evans and Sels charts, (Fig. 165 and Fig. 166).

Synchronous Condensers:

Installation at B: 3 – 30,000 kv.-a. synchronous condensers.
Installation at N: 1 – 28,000 kv.-a. synchronous condenser.
Performance given by the V-curves (Fig. 186).

Load at B:

90,000 kw. at 90 per cent power factor (lagging).

Solution

Following the procedure outlined in Example 6 gives

a.

$$P_A = 127{,}000 \text{ kw.} \qquad Q_A = -11{,}500 \text{ kv.-a.}$$

$$\angle\frac{E_M}{E_N} = 43.5 \text{ deg.}$$

$$P_B = 117{,}000 \text{ kw.} \qquad Q_B = +38{,}000 \text{ kv.-a.}$$

$$P_{LB} = 90{,}000 \text{ kw.} \qquad Q_{LB} = -43{,}600 \text{ kv.-a.}$$

$$P_M = 117{,}000 + 60{,}000 - 90{,}000 = 87{,}000 \text{ kw.}$$

$$Q_M = +13{,}000 \text{ kv.-a.} \qquad \angle\frac{E_M}{E_N} = 28.5 \text{ deg.}$$

$$P_N = 82{,}000 \text{ kw.} \qquad Q_N = +5{,}500 \text{ kv.-a.}$$

Load at N

$$L_N = 82{,}000 \text{ kw.}$$

b. Hydro generators supply

Power $P_{GA} = 127{,}000$ kw.
Reactive power $Q_{GA} = -11{,}500$ kv.-a.

In per cent of the total generator installation, these figures become

$$P_{GA} = \frac{127{,}000 \times 100}{128{,}000} = 99.2 \text{ per cent}$$

and

$$Q_{GA} = -\frac{11{,}500 \times 100}{128{,}000} = -9.0 \text{ per cent}$$

Their field current is

$$I_{f(GA)} = 146 \text{ per cent of normal}$$

c. Synchronous condenser at B takes

$$Q_{CB} = 38{,}000 + 43{,}600 - 13{,}000 = +68{,}600 \text{ kv.-a.}$$

In per cent of the installation this becomes

$$\frac{68{,}600 \times 100}{90{,}000} = 76.3 \text{ per cent}$$

and the field current

$$I_{f(CB)} \doteq 88.8 \text{ per cent of normal}$$

Synchronous condenser at N takes

$$Q_{CN} = +5{,}500 \text{ kv.-a.}$$

or in per cent

$$\frac{5{,}500 \times 100}{28{,}000} = 19.6 \text{ per cent}$$

and the field current becomes

$$I_{f(CN)} = 61.2 \text{ per cent of normal.}$$

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